GEOMETRY WITH GEOGEBRA

PART ONE: TRIANGLES

Notations

AB **distance** between the points A and B

(AB) **line** through the points A and B

$[AB]$ **segment** between the two points A and B (A and B included)

$]AB[$ **segment** between the two points A and B (A and B excluded)

$[AB)$ **ray** or **half-line** with A as initial point and passing through B (i.e. all points P of (AB) such that A is *not* between B and P)

$\widehat{AOB}$ **angle** with **vertex** O and **sides** $[OA)$ and $[OB)$

$M' = p_{(AB)}(M)$ means that M' is the perpendicular projection of M onto the line (AB), i.e. $M' \in (AB)$ and $(MM') \perp (AB)$

In order to keep **macros** defined in an exercise, first save the current figure, then erase this figure in order to keep a blank sheet with the defined macros which you can use for the following exercise.

1) DEFINITION OF SOME MACROS.

Create, on the same sheet, three « macros » to draw, from three points as *input objects*:

a) The **circumscribed circle** of the triangle defined by these three points and its **circumcenter**.

b) The **orthocenter** of the triangle defined by these three points.

c) The **centroid** (or **geometric barycenter**) of the triangle defined by these three points.

2) INCIRCLE AND EXCIRCLES OF A TRIANGLE

a) Angle bisectors

- Draw two lines (AB) and (CD) intersecting at a point O such that O is between A and B and between C and D.
- Construct the bisectors d_1 of $\widehat{AOD}$, d_2 of $\widehat{DOB}$, d_3 of $\widehat{BOC}$ and d_4 of $\widehat{COA}$. What can you say about these four bisectors?
- What can you say about a point $P \in d_1$ with regard to the sides of the angle?
- Draw the tangent circle to (OA) whose centre is P. What can you say about this circle with regard to (OD)? *(save as ex2a and erase the figure)*

b) Give a short justification of why the three inner bisectors of the angles of a triangle $\Delta(ABC)$ are concurrent in one point which is the center, called **incentre,** of a circle called **incircle** of the triangle (make a construction). What about the position of this circle with regard to the sides of the triangle? Create a macro to draw, with the vertices A, B, C as input objects, the incircle of the triangle, its incenter and the three contact points with the sides of the triangle.

c) There exist three circles <u>outside</u> the triangle, called **excircles** of the triangle, which are tangent to (AB), (AC) and (BC). Draw them and create a macro to draw, with the vertices A, B, C as input objects, the excircles of the triangle, their centers and their contact points with the lines (AB), (AC) and (BC). *(save as ex 2b and erase the figure)*

d) Let J_A, J_B, J_C be the centers of the excircles (with J_A opposite to A, etc) and I the center of the incircle of $\Delta(ABC)$. What can you say about the triangle $\Delta(J_AJ_BJ_C)$? What represents I for this triangle?

e) On every side of the triangle you have a contact point with the incircle and a contact point with an excircle. What can you say about the position of these two points?

f) Calculate $r_A + r_B + r_C - r$ where r_A, r_B, r_C are the radii of the three excircles and r the radius of the incircle and compare the result with the radius R of the circumscribed circle. Write a formula involving these five radii. *(save as ex 2c)*

3) TWO PROPERTIES OF THE ORTHOCENTRE

a) Construct:

- a triangle $\Delta(ABC)$ and its orthocenter H
- the triangle $\Delta(DEF)$ such as $(DE)\parallel(AB)$ and $C\in(DE)$, $(EF)\parallel(BC)$ and $A\in(EF)$, $(DF)\parallel(AC)$ and $B\in(DF)$
- the feet A', B' and C' of the altitudes issued of A, B and C respectively (Notice that $A'\in(BC)$, $B'\in(AC)$, $C'\in(AB)$ and <u>not</u> $A'\in[BC]$, etc!)

b) Under what condition can you speak about the *triangle* $\Delta(A'B'C')$?

Definition: The triangle $\Delta(A'B'C')$ is called the **orthic triangle** of $\Delta(ABC)$.

Create a macro to draw the orthic triangle $\Delta(A'B'C')$ (and its vertices) with the points A, B, C as input objects.

c) What can you say about H with regard to the triangle $\Delta(DEF)$?

d) What can you say about the incentre of the orthic triangle?

4) THE EULER LINE (Leonhard Euler, 1707 – 1783, famous Swiss mathematician)

Construct a triangle $\Delta(ABC)$, its circumcenter O, its incenter I, its orthocenter H and its centroid G.

a) What can you say about the positions of these points? Formulate the "theorem of the Euler line".

b) Create a macro to draw the Euler line of a triangle $\Delta(ABC)$ with the points A, B, C as input objects.

5) THE SCHIFFLER POINT (Kurt Schiffler, 1896 – 1986, German manufacturer and amateur mathematician, published his discovery in 1985)

Construct a triangle $\Delta(ABC)$, its incircle, the incenter I and the Euler lines of the four triangles $\Delta(ABC)$, $\Delta(ABI)$, $\Delta(AIC)$ and $\Delta(IBC)$. Formulate the "theorem of the Schiffler point".

6) THE EULER CIRCLE (or « nine-point circle »)

a) Construct:

- a not right-angled triangle $\Delta(\mathrm{ABC})$
- the orthocenter H of $\Delta(\mathrm{ABC})$
- the orthic triangle $\Delta(\mathrm{A'B'C'})$
- the circumscribed circle CC of $\Delta(\mathrm{ABC})$ and it's circumcenter O
- the circumscribed circle CE of $\Delta(\mathrm{A'B'C'})$ and it's center Ω

b) The circle CE meets six segments in their midpoints: find them! Why is this circle also called the "nine-point circle"? Formulate the "theorem of Euler's circle".

c) Create a tool which draws the Euler's circle of a triangle and it's center with the three vertices of the triangle as input objects.

d) Properties of the Euler's circle

- Where is situated exactly the center Ω of CE?
- What can one say about the radius of CE?
- If $\mathrm{P} \in \mathrm{CC}$, what can one say about the segment $[\mathrm{HP}]$?
- Let be C1 the incircle and C2, C3, C4 the three excircles of $\Delta(\mathrm{ABC})$. What can one say about their position compared to CE? Formulate this result which is known as "Feuerbach theorem".

7) THE COMPLETE QUADRILATERAL

a) Construct two secant segments $[\mathrm{AB}]$ and $[\mathrm{AC}]$, two points $\mathrm{D} \in]\mathrm{AB}[$, $\mathrm{E} \in]\mathrm{AC}[$, the segments $[\mathrm{CD}]$, $[\mathrm{BE}]$ and the point $\mathrm{F} \in [\mathrm{CD}] \cap [\mathrm{BE}]$. The figure ABCDEF is called a **complete quadrilateral (or quadrangle)**, the six points A, B, C, D, E, F are called the **vertices** and the three segments $[\mathrm{BC}]$, $[\mathrm{DE}]$ and $[\mathrm{AF}]$ (formed by the three pairs of vertices *not* yet connected) are called the **diagonals** of the complete quadrilateral.

b) The Newton line of a complete quadrangle

This line concerns the three diagonals of the complete quadrangle. Try to find out how! Formulate the "theorem of the Newton line".

c) The Miquel point and the Miquel circle (Auguste Miquel, French m., 19th c.)

Construct the circumcircles of the four triangles which you see on the figure (without the diagonals). What can you say about these circles? What about the centers of these circles? Formulate the "theorem of Miquel's point and Miquel's circle of a complete quadrangle". *(save as Ex7a and erase the figure)*

d) Three cross ratios

Construct a complete quadrilateral ABCDEF and the points $P \in (AF) \cap (DE)$ and $Q \in (AF) \cap (BC)$ (the points A, P, F, Q are collinear). Calculate $\frac{PA}{PF} : \frac{QA}{QF}$, this number is called the **cross ratio of (A, F ;P, Q)**.

If the lines (DE) and (BC) are not parallel there exist two other cross ratios on the figure with a special result. Try to find them!

8) GERGONNE POINT, NAGEL POINT AND MIDDLESPOINT

(Joseph Gergonne, 1771 – 1859, French mathematician, Christian von Nagel, 1803 – 1882, German mathematician)

a) Construct :

- a triangle $\Delta(ABC)$
- the midpoints A', B', C' of $[BC]$, $[AC]$ and $[AB]$ respectively
- the incircle with its center I an its contact points $A_1 \in [BC]$, $B_1 \in [AC]$ and $C_1 \in [AB]$
- the three excircles, their centers I_A, I_B, I_C (where I_A is in front of A, etc.) and their contact points with the sides of $\Delta(ABC)$: $A_2 \in [BC]$, $B_2 \in [AC]$ and $C_2 \in [AB]$
- the lines (AA_1), (BB_1) and (CC_1) who are concurrent in J, called the **Gergonne point** of $\Delta(ABC)$
- the lines (AA_2), (BB_2) and (CC_2) who are concurrent in N, called the **Nagel point** of $\Delta(ABC)$

- the lines $(I_A A')$, $(I_B B')$ and $(I_C C')$ who are concurrent in M, called the **middlespoint** of $\Delta(ABC)$
- Define three macros to draw the Gergonne point, the Nagel point and the middlespoint with A, B, C as input objects *(save as ex8a and erase the figure)*

b) Construct a triangle $\Delta(ABC)$, its Gergonne point J, its Nagel point N, its middlespoint M, its circumcenter O, its incenter I, its centroid G and its orthocentre H. Examine the relative positions of all these points.

c) What represents N for the triangle $\Delta(DEF)$ defined in ex 3a?*(save as ex8b)*

9) THE TAYLOR CIRCLE

a) Construct :

- a triangle $\Delta(ABC)$ and its orthic triangle $\Delta(A'B'C')$ *(cf exercise 3b)*
- $M = p_{(AB)}(A')$
- $N = p_{(AC)}(A')$
- $P = p_{(BA)}(B')$
- $Q = p_{(BC)}(B')$
- $R = p_{(CA)}(C')$
- $S = p_{(CB)}(C')$

b) In this context one speaks about the "Taylor circle" of the triangle $\Delta(ABC)$. Which circle do you think we are talking about? Construct the Taylor circle and its center O.

c) ***Definition***

*The **medial triangle** of a triangleΔ(PQR) is the triangle whose vertices are the midpoints of the sides of the triangle Δ(PQR).*

Construct the medial triangle $\Delta(A''B''C'')$ of the orthic triangle $\Delta(A'B'C')$ of $\Delta(ABC)$ and try to find out the relation between O and $\Delta(A''B''C'')$.

d) Formulate the theorem of the Taylor circle.

10) THE SIMSON AND THE STEINER LINES (Robert Simson, 1687 – 1768, Scottish mathematician, Jakob Steiner, 1796 – 1863, Swiss mathematician)

a) The Simson line

- Construct a triangle $\Delta(ABC)$, its circumscribed circle CC with centre O, a point $P \in CC$ and its perpendicular projections $R = p_{(AB)}(P)$, $S = p_{(AC)}(P)$, $T = p_{(BC)}(P)$. Which line do you think is called the **"Simson line associated to P"**?
- Make the same construction with a point P' which is *not* on the circle CC. What can you deduce from this?
- Formulate the "theorem of the Simson line associated to a point P ".
- Create a macro to draw the Simson line associated to a point P with A, B, C, CC and P as input objects.

b) The Steiner line

- Construct the reflected points U, V, W of P over the lines (AB), (AC) and (BC) respectively: $U = s_{(AB)}(P)$, $V = s_{(AC)}(P)$, $W = s_{(BC)}(P)$. Which line do you think is called the **"Steiner line associated to P"**?
- Make the same construction with the point P' which is *not* on the circle CC. What can you deduce from this?
- Formulate the "theorem of the Steiner line associated to a point P".
- Create a macro to draw the Steiner line associated to a point P with A, B, C, CC and P as input objects. *(Save the figure as Ex10a and erase it)*

c) Properties of the Simson and the Steiner lines

Property 1 :

By turning the point P around the circle CC you can observe that the Steiner lines of the points of CC are all concurrent in a fixed point. Try to recognize this well-known point!

Property 2 :

Let A', B', C' be diametrically opposite to A, B, C respectively. What can you say about the Simson lines associated to A, B, C, A', B', C'?

Property 3 :

Examine the relative positions of the Simson and the Steiner lines associated to the same point. Be as precise as possible!

Property 4 :

Let p and r be the Simson lines associated to the points P and R respectively, then:

- One of the angles formed by p and r is half of the angle …………….
- $p \perp r \Leftrightarrow \cdots$ Does this work for the Steiner lines too? (*Save the figure as Ex10b and erase it).*

Property 5 :

Construct:

- a circle with four points A, B, C, D on it
- the Simson line d_1 associated to A in relation to the triangle $\Delta(BCD)$
- the Simson line d_2 associated to B in relation to the triangle $\Delta(ACD)$
- the Simson line d_3 associated to C in relation to the triangle $\Delta(ABD)$
- the Simson line d_4 associated to D in relation to the triangle $\Delta(ABC)$

What do you observe? (*Save the figure as Ex10c and erase it)*

Property 6 :

Let $\Delta(ABC)$ and $\Delta(DEF)$ be two triangles with the same circumscribed circle CC, a point $P \in CC$, d_1 the Simson line associated to P in relation to triangle $\Delta(ABC)$ and d_2 the Simson line associated to P in relation to triangle $\Delta(DEF)$. What can one say about the angle formed by the two Simson lines? (*Save the figure as Ex10d and erase it)*

11) THE TORRICELLI or FERMAT POINT (Evangelista Torricelli, Italian mathematician, 1608 – 1647, Pierre de Fermat, French m., 1601 – 1665)

a) Definition

- Construct a triangle $\Delta(ABC)$ and three equilateral triangles $\Delta(ABC')$, $\Delta(AB'C)$, $\Delta(A'BC)$ such that A', B' and C' are outside of $\Delta(ABC)$. Three lines, each defined by two points of this figure, are concurrent in one point T called **the Torricelli or Fermat point** of $\Delta(ABC)$: find them!
- This point could also be defined as the intersection point of three circles which are.…..
- The three concurrent lines have two other nice properties….

b) Create a **macro** to draw the Torricelli point of a triangle $\Delta(ABC)$ with A, B, C as input objects, save the figure as *Ex12a)* and erase it before continuing.

c) The position of the Torricelli point.

- Draw a new triangle $\Delta(ABC)$ and its Torricelli point T
- measure the angles of the triangle
- to what condition is T situated on one of the vertices?
- to what condition is T situated outside the triangle?
- to what condition is T situated on $]A,B[$, $]A,C[$ or $]C,B[$?

d) A minimal sum of distances

- For any point M let $f(M) = MA + MB + MC \in \mathbb{R}_+^*$ be the sum of the distances from M to the three vertices of the triangle
- Calculate $f(M)$, $f(T)$, $f(A)$, $f(B)$ and $f(C)$
- For what point is this sum minimal?

Save as *Ex 12b).*

12) THEOREM OF MENELAUS OF ALEXANDRIA (Greek m., 70 – 140)

- Make a construction to complete and verify following theorem:

  Let $\Delta(ABC)$ be a triangle and three <u>collinear</u> points D, E, F, all of them different from A, B and C so that $D \in (AB)$, $E \in (BC)$ and $F \in (AC)$.

  Then $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = \ldots$

- Formulate the converse of this theorem and make a construction to verify it.