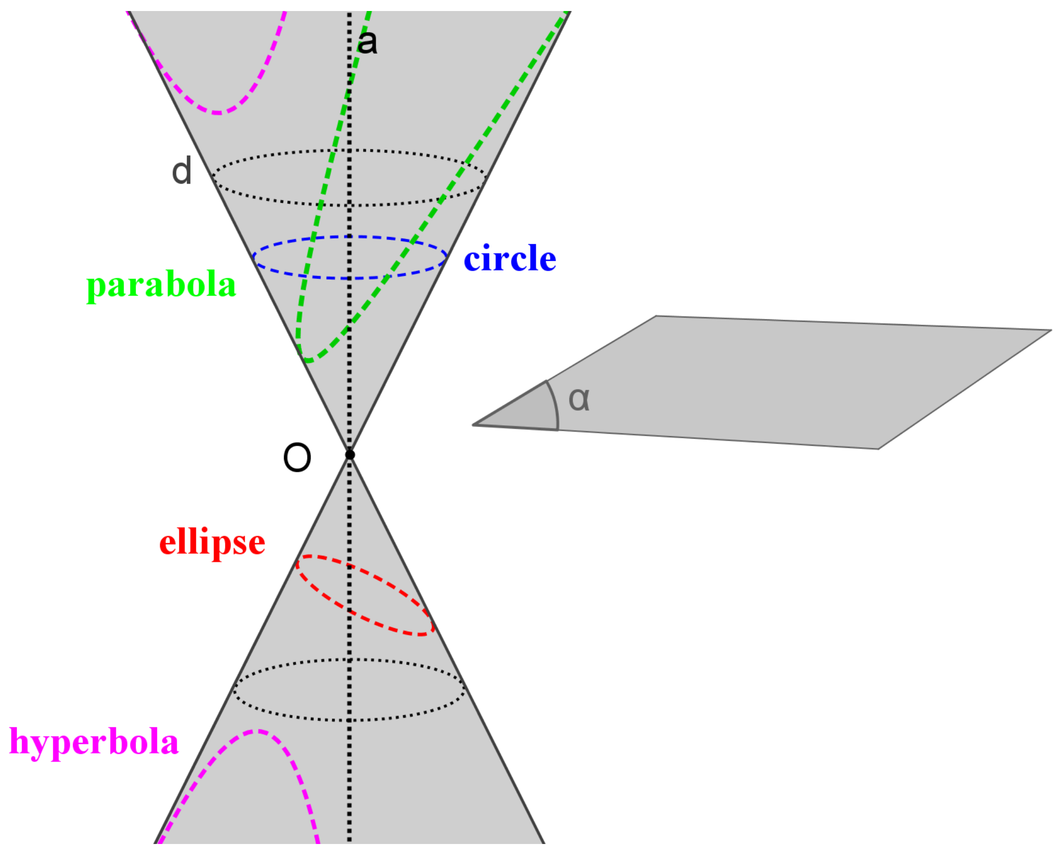
PART TWO: CONIC SECTIONS

Introduction

A **conic section** is a curve you get by intersection of a plane α and a *double infinite cone* which is obtained by rotation (in space!) of a line *d* around an axis *a* (lines *a* and *d* are secant at O):



Depending on the angle in which the plane cuts the cone you get a circle, an ellipse, a parabola or a hyperbola but you can also get a line, two secant lines or even just the point O which are not very interesting and are therefore called “degenerate” conics.

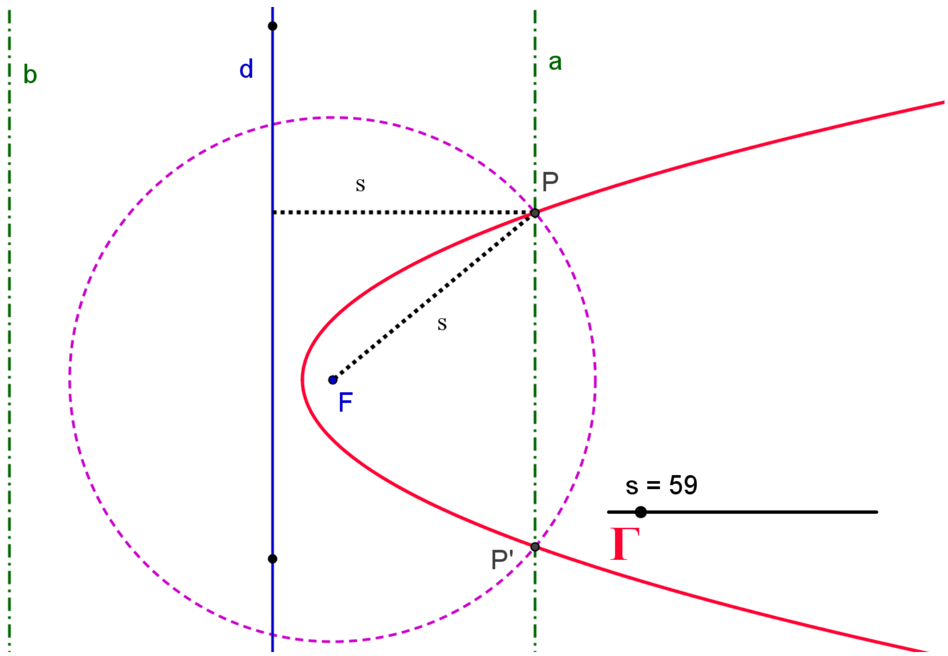
This definition, which explains the term “conic section”, is a 3 – dimensional approach of 2 – dimensional curves which is not very convenient. For this reason we shall give now some « 2 – dimensional » approaches of these curves.

1) The locus of equidistant points to a line and a point.

Given a line **d** and a point **F**, let's construct the locus Γ of all points P which are equidistant from F and d, i.e. $PF = Pd$.

Let **s** be a real positive number, then the locus of the points P such that $PF = s$ is the circle with centre **F** and radius **s** and the locus of the points P such that $Pd = s$ is the union of two parallel lines **a** and **b** so that Γ is the intersection of these two loci (for $s \in \mathbb{R}_+^*$).

Make the following *construction*:



- a point **F**, a line **d** and a slider **s** (with $0 < s < 500$ for example)
- the circle with centre **F** and radius **s**
- the lines **a** and **b** whose distance to **d** equals **s**
- hide all points and lines you needed for this construction and who are not shown on the figure above
- the intersection points P and P' of these lines and the circle
- activate the "trace" function on points P and P' and use slider s to visualize Γ
- to get a more smooth curve (as shown on the figure) use the "**locus**" command

Definition

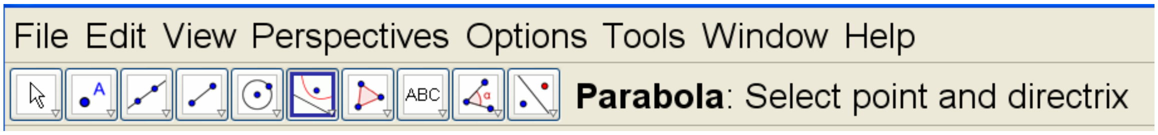
The curve Γ you get is called a **parabola** with **focus F** and **directrix d.**

Questions:

What happens to the parabola if you move F further away from d?

What happens if you put F on the other side of d?

Verify that you get exactly the same curve by using the "**parabola**" command of GeoGebra:



Which line do you think is called the "**principal axis of** Γ?

Which point do you think is called the "**vertex**" of the parabola?

What are the elements of symmetry of a parabola?

2) A monofocal definition of conic sections

- *Definition*

  Given a line **d,** a point **F** and a positive number ε, the locus Γ of all points P such that $PF = \varepsilon \cdot Pd$ is called a **conic section** with **focus** F, **directrix** d and **eccentricity** ε.

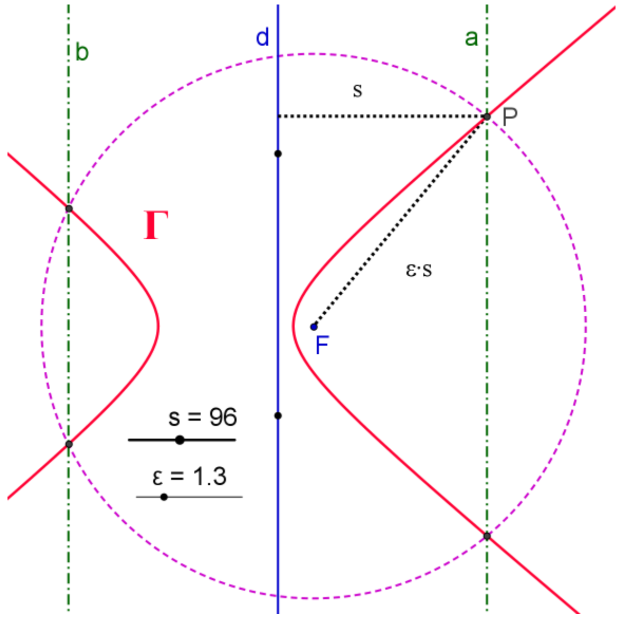
- Note that in this definition we just speak about *one* focus and that's why it's called "monofocal". Later we'll see that ellipses and hyperbolas actually have *two* foci and *two* directices and that these conics can be defined in a completely different way.

- If $\varepsilon = 0$ then $PF = 0 \cdot Pd \Leftrightarrow PF = 0 \Leftrightarrow P = F$ and the locus Γ would be reduced to the single point F! For this reason we assume that $\varepsilon > 0$.

- In exercise 1 we have seen that for $\varepsilon = 1$ Γ is a parabola and we shall examine now the shape of Γ for $\varepsilon > 1$ and for $0 < \varepsilon < 1$.

- *Construction*

  You just have to take again the construction of exercise 1, to add a slider ε (with $0 < \varepsilon < 5$ for example) and to change the radius of the circle in $\varepsilon \cdot s$. Note that for certain values of ε the circle cuts the lines a and b in four points!



- *Definition*

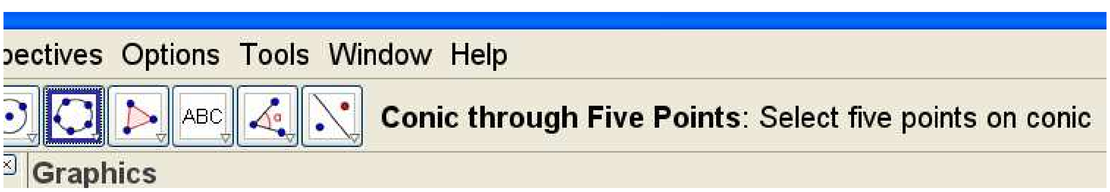
  A conic section with:

  - $0 < \varepsilon < 1$ is called an **ellipse**
  - $\varepsilon = 1$ is called a **parabola**
  - $\varepsilon > 1$ is called a **hyperbola**

- *Questions*

  What can you say about the influence of eccentricity on the shape of Γ? What about the influence of the distance Fd on this shape?

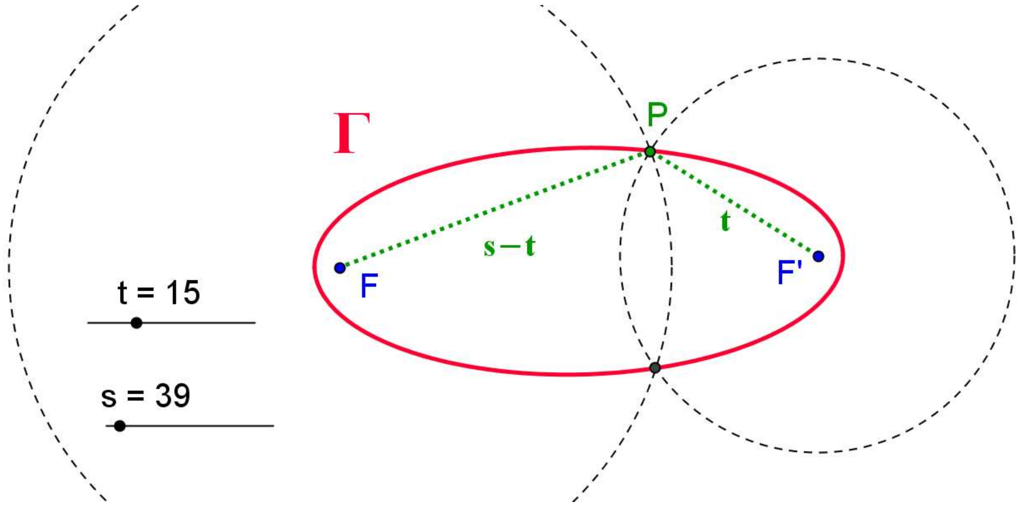
- Draw five points on Γ and then use the command "**conic through five points**" of GeoGebra to verify that the curve you get coincides exactly with Γ:



3) A bifocal definition of an ellipse

Given two points **F** and **F'** and a real number $s > FF'$, let's construct the locus Γ of all points P such that $PF + PF' = s$.

On the following figure it's easily seen how to build this locus….



…. and that Γ is an ellipse but this time defined by two points F and F' called **foci** and a real number **s**.

Verify that you get exactly the same curve by using the "**ellipse**" command of GeoGebra:

Questions

What are some obvious limitations on the numbers s and t? What can you say about the influence of the number s and the distance FF' on the shape of Γ?

Problem

In exercise 2 we have seen that an ellipse is defined by a focus, a directrix and its eccentricity (the "monofocal" definition) and now we see that it can also be defined by two foci and a number s (the "bifocal" definition). We shall now examine the **relation between these two definitions**.

Construct:

- an ellipse Γ with foci F and F' by using the "**ellipse**" command of GeoGebra
- the line $m = (FF')$ which is called the **focal or major axis** of Γ
- the midpoint O of $[FF']$ which is called the **centre** of Γ
- the intersection points V_1 and V_2 of (FF') and Γ

- the perpendicular bisector **m'** of $[FF']$ which is called the **minor axis** of Γ
- the intersection points V_3 and V_4 of the minor axis and Γ
- the points V_1, V_2, V_3 and V_4 are called the **vertices** of Γ
- the measures $a = OV_1 = OV_2$, $b = OV_3 = OV_4$ and $c = OF = OF'$: the **distances** $2a = V_1V_2$ and $2b = V_3V_4$ are also called respectively **major axis** and **minor axis** of Γ and the distance $2c = FF'$ is called the **focal distance** of Γ
- the numbers $e = \dfrac{c}{a}$ and $f = \dfrac{a^2}{c}$
- the two lines d and d' such that $d \perp (FF')$ and $d' \perp (FF')$ (d is the closest to F, d' the closest to F') and $Od = Od' = f$
- a point $M \in \Gamma$
- the measures MF, Md and the number $\dfrac{MF}{Md}$
- the measures MF', Md' and the number $\dfrac{MF'}{Md'}$
- the number $\dfrac{MF + MF'}{2}$

Question

What can you conclude from all this?

4) A bifocal definition of an hyperbola

Given two points **F** and **F'** and a real number **s**, let's construct the locus Γ of all points P such that $|PF - PF'| = s$.

Let's notice that for $s = 0$ we get $PF = PF'$ and Γ is the perpendicular bisector **m** of $[FF']$ so that from now on we assume that $s > 0$.

$$|PF - PF'| = s \Leftrightarrow \begin{cases} PF - PF' = s & \text{if } PF > PF' \\ PF' - PF = s & \text{if } PF' > PF \end{cases}$$

$$\Leftrightarrow \begin{cases} PF - PF' = s & \text{if P on the same side of m than F'} \\ PF' - PF = s & \text{if P on the same side of m than F} \end{cases}$$

<u>case 1</u>: $PF' > PF$

Let's define $PF = t$ and $PF' = s + t$ then $PF' - PF = s$ and it comes :

$PF' \le PF + FF' \Leftrightarrow s + t \le t + FF' \Leftrightarrow s \le FF'$,

$FF' \le FP + PF' \Leftrightarrow FF' \le t + s + t \Leftrightarrow FF' - s \le 2t \Leftrightarrow FF'/2 - s/2 \le t$,

and $\Gamma = \mathcal{C}(F,t) \cap \mathcal{C}(F', s+t)$

<u>case 2</u>: $PF > PF'$

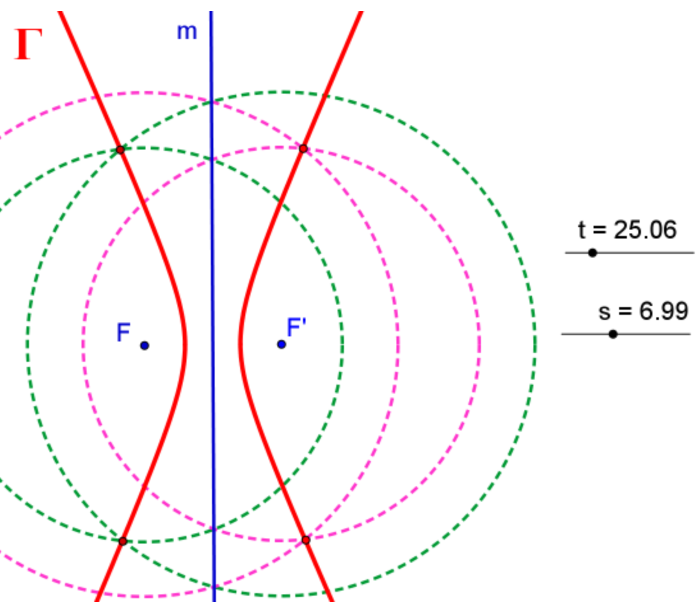
Let's define $PF' = t$ and $PF = s + t$ then $PF - PF' = s$ and it comes :

$PF \le PF' + FF' \Leftrightarrow s + t \le t + FF' \Leftrightarrow s \le FF'$,

$FF' \le FP + PF' \Leftrightarrow FF' \le t + s + t \Leftrightarrow FF' - s \le 2t \Leftrightarrow FF'/2 - s/2 \le t$,

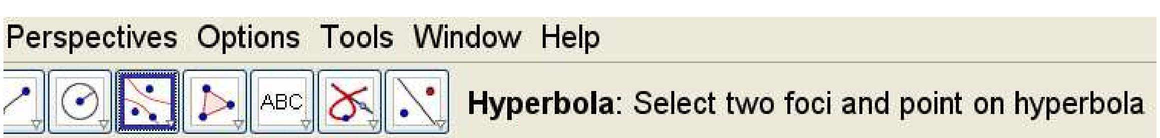
and $\Gamma = \mathcal{C}(F,t) \cap \mathcal{C}(F', s+t)$

On the following figure (where $0 < s < FF'$ and $FF'/2 - s/2 \le t \le 200$) it's easily seen how to build this locus….



…. and that Γ is a hyperbola but this time defined by two points F and F' called **foci** and a real number **s**.

Verify that you get exactly the same curve by using the "**hyperbola**" command of GeoGebra:



Question

What can you say about the influence of the number s and the distance FF' on the shape of Γ?

Problem

In the same way as in exercise 3 we shall now examine the relation between the monofocal definition (based on a focus, a directrix and the eccentricity) and the bifocal definition (based on two foci and a point) of a hyperbola.

Construct:

- a hyperbola Γ with foci F and F' by using the "**hyperbola**" command of GeoGebra
- the line $m = (FF')$ which is called the **focal axis** of Γ
- the midpoint O of $[FF']$ which is called the **centre** of Γ
- the intersection points **V** and **V'** of (FF') and Γ which are called are called the **vertices** of Γ
- the measures $a = OV = OV'$ and $c = OF = OF'$: the distance $2c = FF'$ is called the **focal distance** of Γ
- the numbers $b = \sqrt{c^2 - a^2}$, $e = \frac{c}{a}$ and $f = \frac{a^2}{c}$
- the two lines d and d' such that $Od = Od' = f$, $d \perp (FF')$ and $d' \perp (FF')$ (d is the closest to F, d' the closest to F')
- the rectangle (ABCD) such that m is the perpendicular bisector of $[BC]$ and of $[AD]$ and that $VB = VC = V'A = V'B = b$
- the lines (AC) and (BD)
- a point $M \in \Gamma$
- the measures MF, Md and the number $\frac{MF}{Md}$
- the measures MF', Md' and the number $\frac{MF'}{Md'}$
- the number $\frac{|MF - MF'|}{2}$

Questions

What can you say about the lines (AC) and (BD)? What can you conclude from all this?

5) Algebraic curves of second degree

Definition

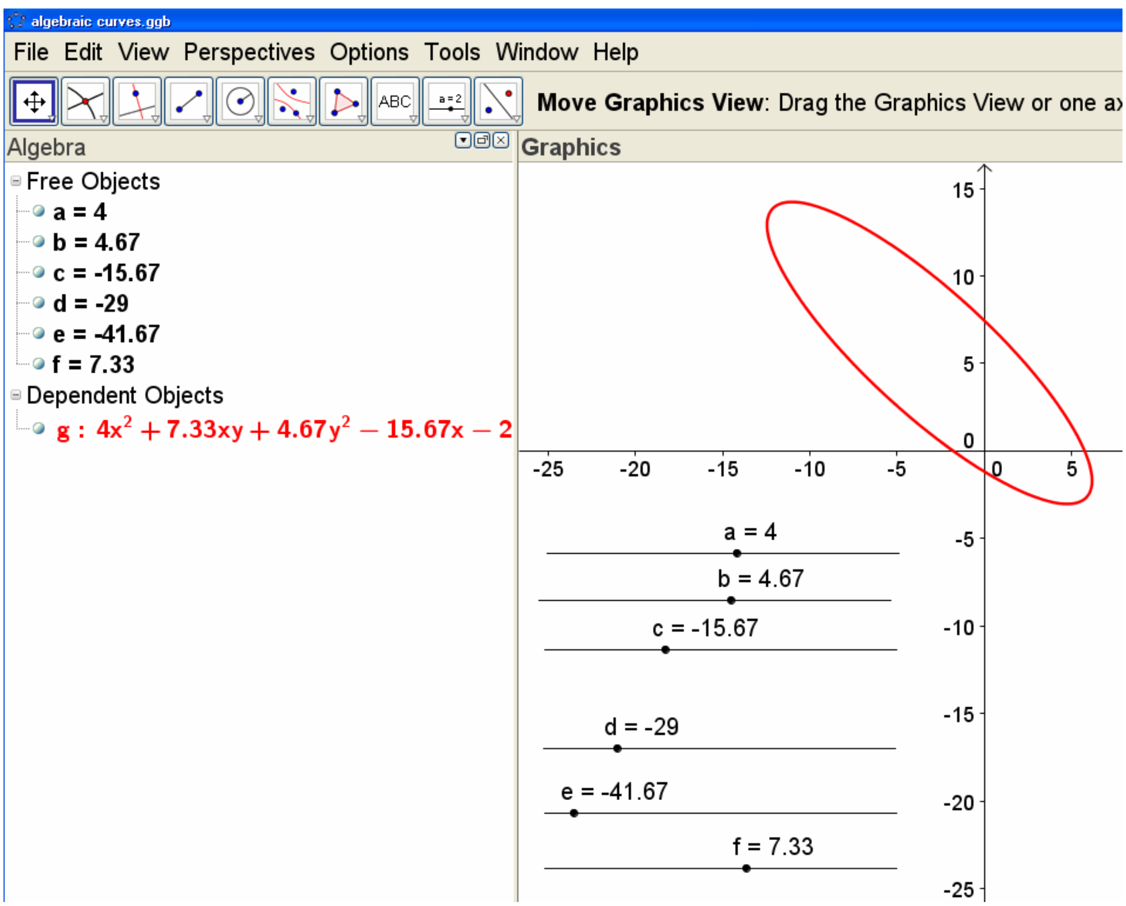
An **algebraic curve of the second degree** is the set of all points $P(x;y)$ whose coordinates x, y verify an equation of the second degree:

$$\boxed{ax^2+by^2+cx+dy+e+fxy=0}$$

where a, b, c, d, e, f are real coefficients.

You can enter such an equation in the “input bar” at the bottom of the screen, for example if you type $x^2+y^2-16=0$ you get a circle of centre $O(0;0)$ and radius 4.

In order to try a lot of different curves it’s more convenient to define first 6 sliders a, b, c, d, e, f (whose values can be fixed for example between –50 and 50) and then enter $a*x^2+b*y^2+c*x+d*y+e+f*x*y=0$ (the * for multiplication are a must!). Now you can vary these coefficients and observe the shapes you get!



Try following examples:

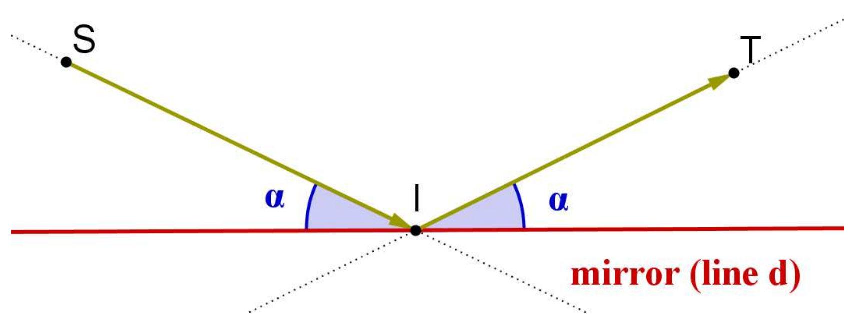
- $a = b = f = 0$ then you get of course a **line** (equation of degree 1)
- $a = b \neq 0$ and $f = 0$ then you get a **circle**, a point (for example the point $(3;-4)$ if $a = b = 1$, $c = -6$, $d = 8$, $e = 25$ and $f = 0$) or **nothing** (for example if $a = b = c = d = e = 1$). "Nothing" means that Γ, the set of points which verify this equation i.e. the curve, is the empty set : $\Gamma = \varnothing$.
- $ab > 0$, $a \neq b$, and $f = 0$ then you get a ……….... or …..…… or …………
- $ab < 0$ and $f = 0$ then you get a ……….... or …………
- $b = d = f = 0$ then you get.............. or …………..
- $a = c = f = 0$ then you get................ or …………..
- $a \neq 0$, $d \neq 0$ and $b = f = 0$ then you get ……….…
- $b \neq 0$, $c \neq 0$ and $a = f = 0$ then you get ………………………
- for all these examples $f = 0$, what happens if $f \neq 0$? …………………………

All the curves you get with this equation are called "conics" but those who are not a parabola, a hyperbola, an ellipse or a circle are said to be "**degenerate conics".**

6) Optical properties of conic sections

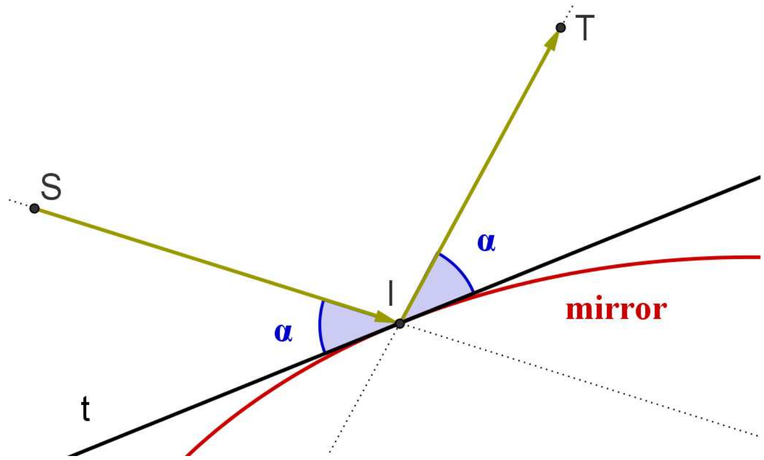
a) Preliminary: a ray of light reflected by a mirror

If a ray of light (or any other kind of wave…), issued from a source point S, meets a mirror (represented by a line d) at the impact point I it is reflected in the direction of the target point T so that the angles formed by $[IS)$ and $[IT)$ with d have the same measure α:



A very simple way to create that figure is to reflect line (SI) in line d using the command “**Reflect Object in Line**”.

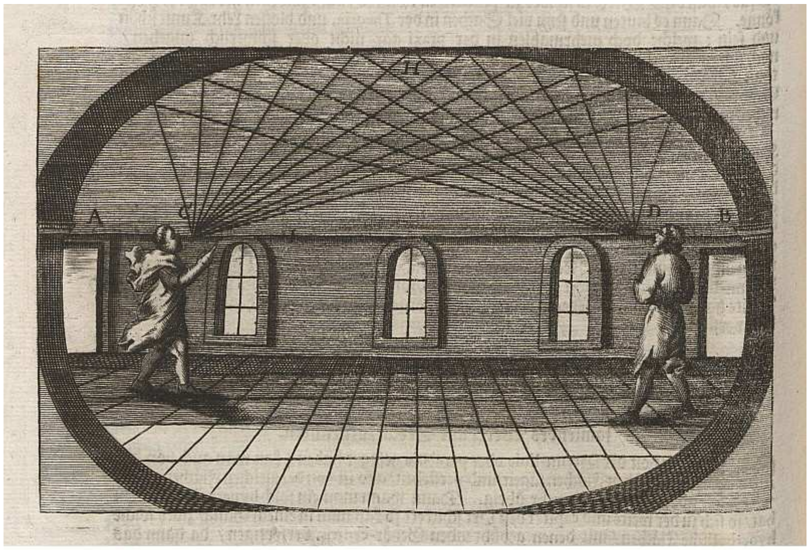
If the mirror is not flat you can replace it symbolically by the tangent line at the point I because in an infinitesimal neighbourhood of I there is no difference between the curve and its tangent at I:



b) An optical property of an ellipse

Given an ellipse Γ (considered as a mirror!), its foci F and F’ and a point I on Γ, make a construction in order to find out what happens to a ray sent out at F and meeting Γ in I! Give a correct formulation of this property!

An application of this property on acoustics is shown in the following picture, a German engraving of 17th century which shows a “whispering gallery”, a very popular subject in architecture of these times.



Quelle: Deutsche Fotothek

c) An optical property of a parabola

Given a parabola Γ (considered as a mirror!), its focus F and a point I on Γ, make a construction in order to find out what happens to a ray sent out at F and meeting Γ in I! Give a correct formulation of this property!

Applications of this property are: headlights for cars, satellite dishes, etc.

d) An optical property of the directrix of a parabola

Construct following figure: a line d, a point F, Γ the parabola of focus F and directrix d, a point $P \in \Gamma$, t the tangent of Γ at P, M the intersection point of t and d and t' the perpendicular to t dropped from M. What can one say about t'? Formulate that property of the directrix of a parabola!

e) An optical property of a hyperbola

Imagine a mirror in the shape of half a hyperbola (with foci F and F') and a source of light S outside the mirror. What happens if you send a ray of light from S in the direction of the focus F which it can't reach because it is behind the mirror?