Sets, probabilities and combinatorics.

1) Definitions, notations, examples

- A **set** is a well defined collection of distinct objects called **elements** of the set. A set is usually denoted by a Latin or a Greek capital letter: A, B, C, Ω, Γ, …

  Remarks

  - "well defined" means that it must be completely clear for any object if it is or not an element of a given set. For example you can't speak of the set of all intelligent people or of the set of all big numbers.
  - "distinct" means that you can't put the same element more than once in a set. For example you can't say that E is the set of the elements 3, 5, 3, and 2.

- A set can be defined by:
  - a **written description**.

    e.g. E is the set of all even integers greater than 7 and smaller than or equal to 18
  - a **list or roster** of its elements.

    e.g. $E = \{8; 10; 12; 14; 16; 18\}$
  - the **set-builder notation**.

    e.g. $E = \{x \,/\, x \text{ is an integer and } 7 < x \leq 18\}$

    which is read: "E is the set of all x **such that** x is…."

- The <u>sets of numbers</u> have the following special notations:

  $\mathbb{N} = \{x \,/\, x \text{ is a natural number (i.e. a positive integer)}\} = \{0; 1; 2; 3; 4; \cdots\}$

  $\mathbb{Z} = \{x \,/\, x \text{ is an integer (positive or negative)}\} = \{\cdots -3; -2; -1; 0; 1; 2; 3 \cdots\}$

  $\mathbb{Q} = \{x \,/\, x \text{ is a rational number}\} = \left\{\frac{a}{b} \,/\, \text{ a and b are integers and } b \neq 0\right\}$

  $\mathbb{R} = \{x \,/\, x \text{ is a real number (i.e. rationnal or irrationnal)}\} = \left\{-5; \frac{3}{7}; -8{,}91; -\sqrt{17}; \sqrt{3}; \pi; \cdots\right\}$

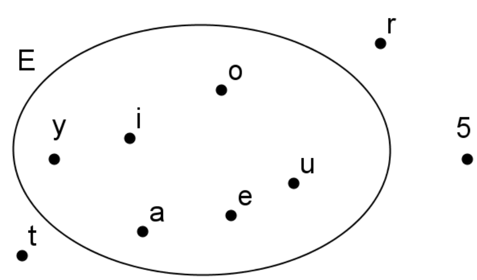
  $\mathbb{N}^*, \mathbb{Z}^*, \mathbb{Q}^*, \mathbb{R}^*$ are the same sets without 0!

- The symbols $\in$ and $\notin$: these symbols are placed <u>between an element and a set</u> to express that this element belongs ($\in$) or not ($\notin$) to this set.
  e.g. $10 \in E$, $11 \notin E$
- The **cardinality** or **cardinal number** of a <u>finite</u> set E is the number *n* of elements of E. You can find various notations for this:

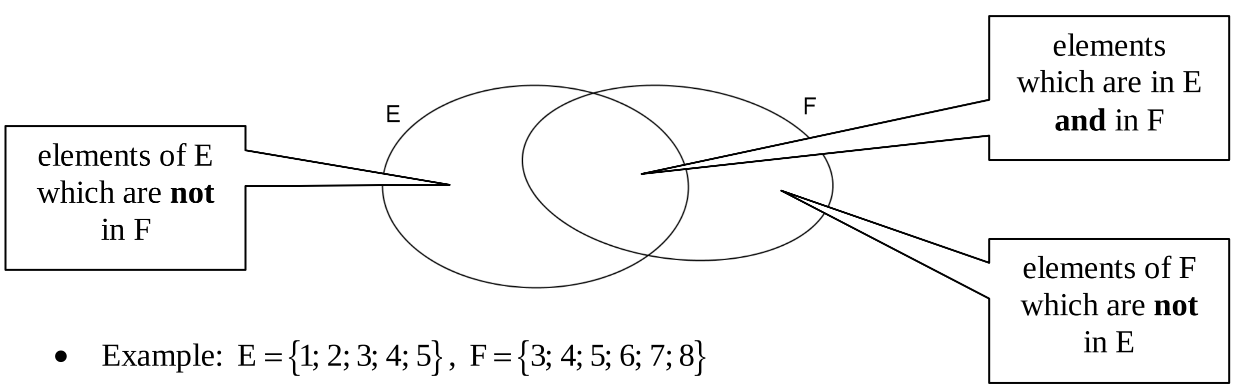
$$|E| = n \text{ or } n(E) = n \text{ or } \text{Card}(E) = n \text{ or } \#E = n$$

2) Venn diagrams

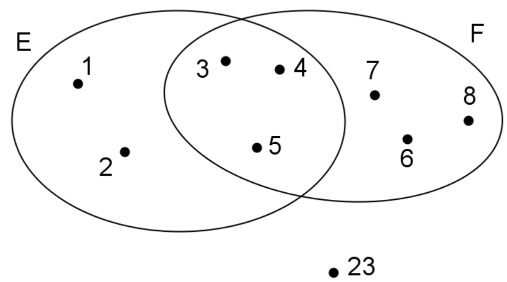
- A **Venn diagram** of a set E is a closed line inside which you put the elements of E and outside the elements which do not belong to E.
- Example: $E = \{i, o, e, u, a, y\}$



- You can draw two sets on the same Venn diagram:

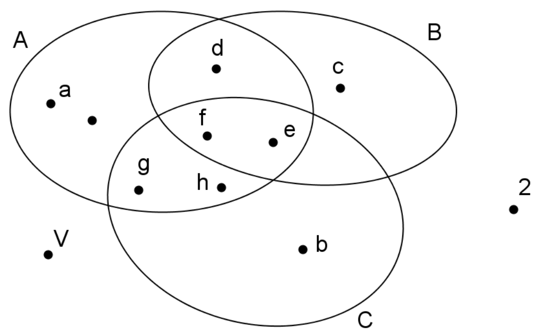


- Example: $E = \{1; 2; 3; 4; 5\}$, $F = \{3; 4; 5; 6; 7; 8\}$



- Three sets are also possible, for example:

  $A=\{d;e;f;g;h\}$, $B=\{a;c;d;e;f\}$, $C=\{b;e;f;g;h\}$



3) Equal sets and subsets

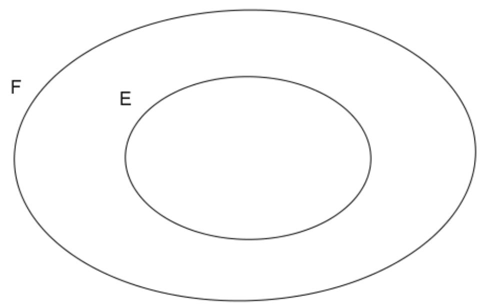
- Two sets are **equal** iff they have the same elements, in any order.

  e.g. $A=\{1;2;3;4;5\}$, $B=\{4;1;3;5;2\}$, $C=\{2;3;4;5;7\}$

  $A=B$ but $A\neq C$

- The only set that has no element is called the **empty set** or **null-set** and is denoted: $\{\ \}$ or $\varnothing$

- A set E is said to be a **subset** of a set F if all the elements of E are also elements of F. Notation: $E\subset F$. On a Venn diagram you can draw E entirely inside F:



- It's obvious that for any set E: $E\subset E$ and $\varnothing\subset E$! Every subset which is different from E and $\varnothing$ is called a **proper subset**.

- If there is at least one element of E which is not in F, E is not a subset of F. Notation: $E\not\subset F$

- Example: $A=\{1;2;3;4;5\}$

  $\{2;4;5\}\subseteq A$, $\{1\}\subseteq A$, $\{2;4;5\}\subseteq A$, $\{1;2;3;4;5\}\subset A$, $\{2;7;5\}\not\subset A$

- The **power set** of any set E is <u>the set of all subsets</u> of R, denoted $\mathcal{P}(R)$.

  Example: $R=\{a;b\}$, then $\mathcal{P}(R)=\{\varnothing;\{a\};\{b\};R\}$ and $|\mathcal{P}(R)|=4$

  So the sets $\varnothing$, $\{a\}$, … are *elements* of the set $\mathcal{P}(R)$:

  $\varnothing\subset R$ but $\varnothing\in\mathcal{P}(R)$, $\{a\}\subset R$ but $\{a\}\in\mathcal{P}(R)$, etc.

- It's easily shown that if $|R|=n$ then $|\mathcal{P}(R)|=2^n$.

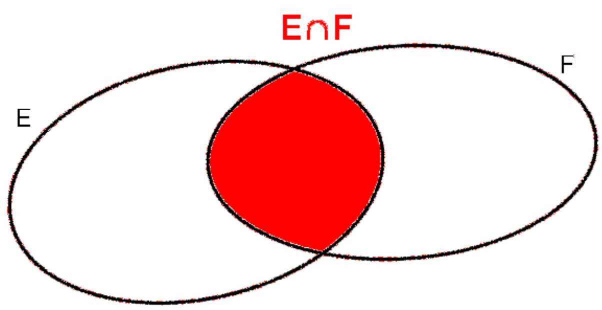
4) Set operations

Let E and f be two sets.

a) Intersection

The **intersection** $E\cap F$ of E and F is the set of all elements that belong to E **and** to F:

$$E\cap F=\{x\,/\,x\in E \text{ and } x\in F\}$$



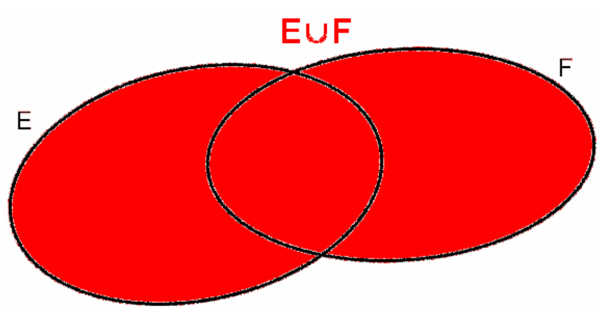
We say that E and F are **disjoint** if $E\cap F=\varnothing$.

Example: $A=\{1;2;3;4;5\}$, $C=\{2;3;4;5;7\}$ then $A\cap C=\{2;3;4;5\}$

b) Union

The **union** $E\cup F$ of E and F is the set of all elements that belong to E **or** to F:

$$E\cup F=\{x\,/\,x\in E \text{ or } x\in F\}$$



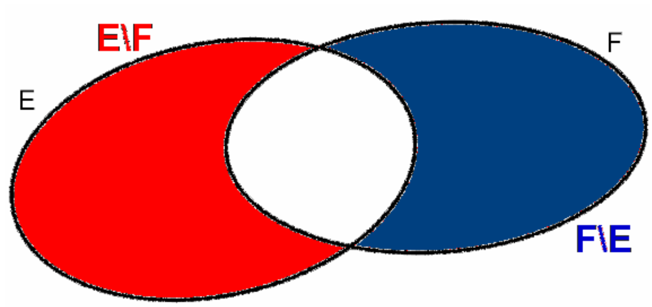
$$|E \cup F| = |E| + |F| - |E \cap F|$$

$$|E \cup F| = |E| + |F| \text{ iff } E \cap F = \varnothing$$

c) Difference

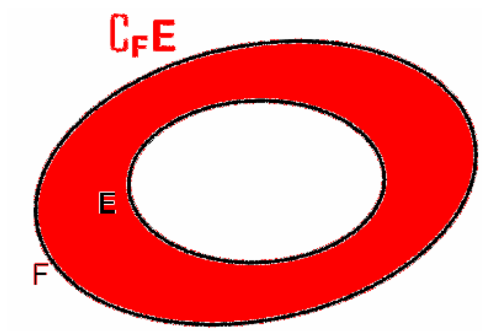
The **difference** $E \setminus F$ of E and F is the set of all elements that belong to E but **not** to F:

$$\boxed{E \setminus F = \{x \,/\, x \in E \text{ and } x \notin F\}}$$



The sets E\F and F\E are disjoint !

If $E \subset F$ the difference E\F is also called **complement** of E in F and noted: $C_F E$.



If all sets are obviously subsets of a same set Ω (as in probability theory) the complement of E is simply denoted $\overline{E}$ which is then considered as the "negation" of E: $\overline{E} = \{x \,/\, x \notin E\}$. In this context you have following properties:

- $\overline{\Omega} = \varnothing$
- $\overline{\varnothing} = \Omega$
- $\forall A, B \in \mathcal{P}(\Omega) \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$ (Morgan law)
- $\forall A, B \in \mathcal{P}(\Omega) \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$ (Morgan law)
- $\forall A \in \mathcal{P}(\Omega) \quad |\overline{A}| + |A| = |\Omega|$

d) Cartesian product

The **Cartesian product** of two sets E and F is the set $E\times F$ of all **ordered pairs** $(x;y)$ where $x\in E$ and $y\in F$:

$$E\times F=\{(x;y)/x\in E \text{ and } y\in F\}$$

As these pairs are **ordered**, $(x;y)\neq(y;x)$ (if $x\neq y$) and so $E\times F\neq F\times E$ in general. However $E\times F=F\times E$ in following two cases:

- if $E=F$, then $E\times F=E\times E$ is also denoted by E^2.
- if at least one of the two sets is the empty set because: $E\times\varnothing=\varnothing\times E=\varnothing$.

Generalization

The Cartesian product of n sets $E_1, E_2, E_3, \ldots, E_n$ is the set $E_1\times E_2\times E_3\times\ldots\times E_n$ of all **ordered n-tuples** $(x_1;x_2;x_3;\ldots;x_n)$ where $\forall i=1,2,\ldots,n \quad x_i\in E_i$:

$$E_1\times E_2\times E_3\times\ldots\times E_n=\{(x_1;x_2;x_3;\ldots;x_n)/\forall i=1,2,\ldots,n \quad x_i\in E_i\}$$

5) Sample space and events (probability theory)

- A **random experiment** is an experiment which has a well-defined set of possible outcomes but whose outcome cannot be predicted. The set of all **possible outcomes** of such an experiment is called **the sample space** and will be denoted by Ω. Each element of the sample space is called a **sample point** or an **elementary event** and corresponds to one possible outcome of the random experiment.
- *Examples*

(1) If we **toss a coin**, there are 2 possible outcomes, *heads* (H) or *tails* (T), so $\Omega=\{H;T\}$ and $|\Omega|=2$.

(2) If we **roll a die**, then $\Omega=\{1;2;3;4;5;6\}$ and $|\Omega|=6$.

(3) If we roll a die twice, then $\Omega=\{(x;y)/x,y=1;2;3;4;5 \text{ or } 6\}=A\times A=A^2$ where $A=\{1;2;3;4;5;6\}$ and $|\Omega|=6^2=36$.

(4) If we **toss the coin 3 times** then $\Omega = \{H,T\}\times\{H,T\}\times\{H,T\} = \{H,T\}^3$ and $|\Omega| = 2^3 = 8$. The elements of Ω are the 8 triplets (H,H,H), (H,H,T), (H,T,H), … (T,T,T). We will often use the *simplified notation*:

$$.\Omega = \{HHH, HHT, HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$

(5) An association of 80 members wants to elect a committee of three members: a president, a secretary and a treasurer. Then

$$\Omega = \{(x;y;z)\ /\ x = \text{the president, } y = \text{the secretary, } z = \text{the treasurer}\}$$

(6) An association of 80 members got 3 theater tickets for free, so they draw 3 members by lots to get the tickets: $\Omega = \{\text{subsets of 5 members out of 80}\}$

We shall see later how to calculate $|\Omega|$ for the examples (5) and (6).

- An **event** is a subset of the sample space Ω. The empty set $(\varnothing \subset \Omega)$ is called the **impossible event** and Ω itself $(\Omega \subset \Omega)$ is called the **certain event**.
- *Examples (the same as above)*

(1) $A = \{H\} \subset \Omega$

(2) B: to get at least 5, $B = \{5;6\} \subset \Omega$

C: to get 9, $C = \varnothing$ (impossible event)

(3) D: to get two times the same result, $D = \{(1;1),(2;2),(3;3),(4;4),(5;5),(6,6)\}$

(4) E: to get two H and one T, $E = \{HHT,\ HTH,\ THH\}$

(5) F: the age of the president must be at least 55 and the treasurer shouldn't be more than 30 years old.

(6) The winners of the tickets have to be member of the association since at least 3 years.

6) Equiprobability

- In a random experiment for which you can "reasonably" admit that all the possible outcomes are equally likely (i.e. have the same chance to occur) you can assign to every elementary event $\omega \in \Omega$ the probability $p(\omega) = \dfrac{1}{|\Omega|}$ which is a real number between 0 and 1. Of course the sum of all $p(\omega)$ for $\omega \in \Omega$ equals 1:

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

- *Examples (the same as above)*

  (1) $p(H) = p(T) = \dfrac{1}{2} = 0{,}5$ which means that if you toss a coin N times (N a very large number!) and you denote n the number of heads you got, then $\dfrac{n}{N} \simeq 0{,}5$, i.e. you get more or less 50% heads when you toss a coin a very large number of times.

  (2) $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \dfrac{1}{6}$

  (3) $\forall (x;y) \in \Omega \quad p((x;y)) = \dfrac{1}{36}$

  (4) $\forall (x;y;z) \in \Omega \quad p((x;y;z)) = \dfrac{1}{8}$

- If A is an event of such a sample space, the probability of A, denoted by $p(A)$, is defined by the Laplace formula as:

$$p(A) = \frac{|A|}{|\Omega|} = \frac{\text{number of favourable cases}}{\text{number of possible cases}}.$$

- Following properties are quite obvious:

  - $p(\Omega) = 1$ and $p(\varnothing) = 0$
  - If $A \neq \Omega$ and $A \neq \varnothing$ then $0 < p(A) < 1$
  - The probability that **A does not occur**, i.e. $p(\overline{A})$, is given by:

    $$p(\overline{A}) = 1 - p(A)$$

    This formula is very useful in case it's easier to count the sample points in $\overline{A}$ than those in A.

- *Examples (the same as above)*

(1) $p(\{H,T\})=\frac{2}{2}=1,\ p(\{H\})=p(\{T\})=\frac{1}{2},\ p(\varnothing)=\frac{0}{2}=0$

(2) $p(B)=\frac{2}{6}=\frac{1}{3}$

(3) $p(D)=\frac{6}{36}=\frac{1}{6}$

(4) $p(E)=\frac{3}{8}$

Exercises 1 - 13

7) Elements of combinatorics

In probability theory counting the number of elements of certain sets (sample space or events) has turned out to be very important. Till now we just met sets with few elements and we didn't have any problems in counting them but what about sets with thousands or millions or even more elements? The purpose of **combinatorics** (or **combinatorial analysis**) is to develop some special techniques of counting.

Most counting problems can be boiled down to drawing by lots of r objects from an urn containing n objects. To distinguish these draws there are **two criteria**:

- Order**:** does the order in which the objects are drawn count or not?
- Repetition or replacement**:** is every object drawn to be put back in the urn before the next object is drawn in order to permit its repetition or not?

So there exist four different types of draws by lots:

- Draws with order and with repetition (OR): they are called **permutations (or arrangements) with repetition** of n objects taken r at a time.
- Draws with order and without repetition ($O\overline{R}$): they are called **permutations (or arrangements)** without repetition of n objects taken r at a time.
- Draws without order and without repetition ($\overline{O}\,\overline{R}$): they are called **combinations** without repetition of n objects taken r at a time.
- Draws without order and without repetition ($\overline{O}\,\overline{R}$): they are called **combinations with repetition** of n objects taken r at a time or **multisets.**

Remark: The term **"permutation"** or **"arrangement"** is used for ordered draws, while the term **"combination"** indicates that order doesn't play any role.

We shall start our study of these different types by the very important special case of the permutations of n objects i.e. permutations without repetition ($O\overline{R}$) of ***n*** objects taken ***n*** at a time.

a) Permutations of n objects

Example

In how many different ways 4 persons a, b, c, d can seat on 4 chairs numbered 1, 2, 3, 4?

- For the 1st place there are 4 possibilities
- For the 2nd place there are 3 possibilities
- For the 3rd place there are 2 possibilities
- For the 4th place there is only 1 possibility
- Total: $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possibilities

Definition

For any integer n you call **n factorial** the number **n!** defined by:

$$\boxed{\begin{array}{l} 0! = 1! = 1 \\ \forall n > 1 \quad n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n \end{array}}$$

Property

n! is the number of **permutations of n objects**, i.e. the number of ways one can arrange these objects in a certain order without repetition.

Example

There are 25! possibilities to place 25 persons on 25 seats. Let's assume that a computer can list 10^9 of these possibilities per second than he would need $\frac{20!}{10^9 \cdot 3600 \cdot 24 \cdot 365,24} \approx 492\,000\,000$ years to list all of these possibilities!

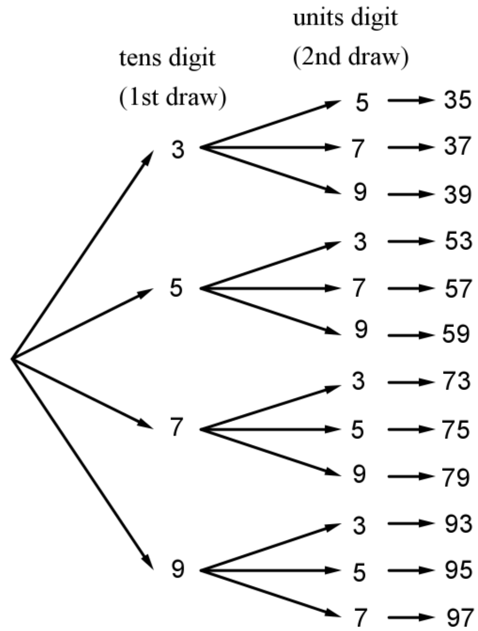
b) Permutations without repetition of n objects taken r at a time

Example

How many two-digit numbers with two different digits chosen from 3, 5, 7, 9 exist?

In other words: how many different draws $O\overline{R}$ *of 2 balls from an urn with 4 balls marked 3, 5, 7, 9 exist?*

These numbers (draws), which may also be considered as ordered pairs $(x;y)$ whose elements are chosen different among the digits 3, 5, 7, 9, are called **permutations (without repetition) of 4 objects taken 2 at a time**. There are $4\cdot 3=12$ such numbers as shown by following diagram:



General case

A **permutation (without repetition) of n objects taken r at a time** (with $r \le n$) is an **r-tuple** $(u_1;u_2;\ldots;u_r)$ whose elements are **all different** and chosen among **n** objects or a draw of r objects with order and without repetition of an urn containing n objects. The number P_n^r of these permutations is given by the formula:

$$P_n^r = n\cdot(n-1)\cdot\ldots\cdot(n-p+1) = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-p+1)\cdot(n-p)\cdot\ldots 2\cdot 1}{(n-p)\cdot\ldots 2\cdot 1} = \frac{n!}{(n-p)!}$$

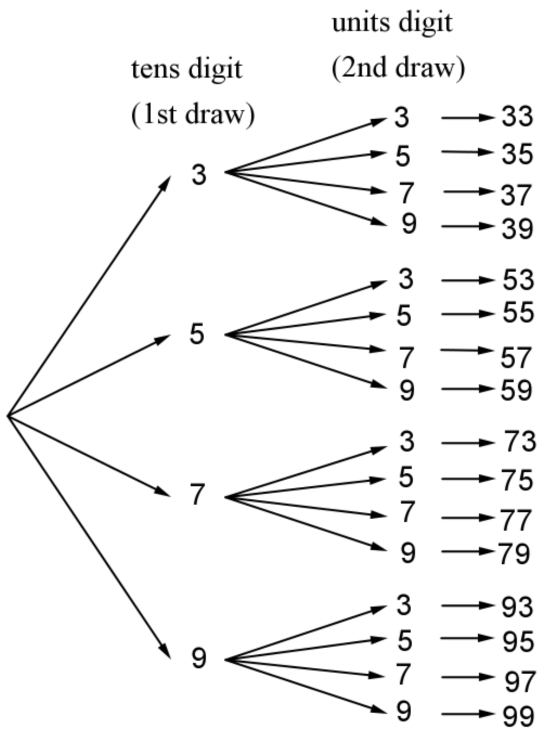
c) Permutations with repetition of n objects taken r at a time

Example

How many two-digit numbers with digits chosen from 3, 5, 7, 9 exist?

In other words: how many different draws OR of 2 balls from an urn with 4 balls marked 3, 5, 7, 9 exist?

These numbers (draws), which may also be considered as ordered pairs $(x;y)$ whose elements are chosen among the digits 3, 5, 7, 9, are called **permutations with repetition of 4 objects taken 2 at a time**. There are $4 \cdot 4 = 4^2 = 16$ such numbers.



General case

A **permutation with repetition of n objects taken r at a time** (with $r \leq n$) is the set of all **r-tuples** $(u_1; u_2; \ldots; u_r)$ whose elements are chosen among the n objects.

The number PR_n^r of these permutations is given by the formula:

$$\boxed{PR_n^r = n^r}$$

d) Combinations (without repetition) of n objects taken r at a time

Example

There are 5 balls in an urn numbered from 1 to 5 and we draw 3 of them without considering the order and without replacement. The result of such a draw is a **subset** of 3 elements of $U = \{1;2;3;4;5\}$, e.g. $\{4;1;2\}$, and is called a **combination of 5 objects taken 3 at a time**. Let's denote the number of these combinations by C_5^3 and let's compare this number with $P_5^3 = \frac{5!}{2!}$ seen previously.

To the subset $\{4;1;2\}$ we can associate 3! permutations: $(1;2;4),(1;4;2),(2;1;4)$ $(2;4;1),(4;1;2),(4;2;1)$ and so we can do with every combination. Thus we get:

$$P_5^3 = 3! \cdot C_5^3 \Leftrightarrow C_5^3 = \frac{P_5^3}{3!} = \frac{5!}{2!3!}$$

General case

A c**combination (without repetition) of n objects taken r at a time** (with $r \le n$) is a **subset** with cardinal r of a set with cardinal n or a draw of r objects without order and without repetition of an urn containing n objects. The number of these combinations, denoted by C_n^r or $\binom{n}{p}$, is given by the formula:

$$\boxed{C_n^r = \frac{n!}{(n-p)!p!}}$$

Examples

- There are $C_{10}^3 = \binom{10}{3} = \frac{10!}{3!7!} = 120$ ways to form a committee of 3 persons from a group of 10 persons.
- Out of a pack of 52 cards you can chose $C_{52}^5 = \binom{52}{5} = \frac{52!}{5!47!} = 2\,598\,960$ different hands of 5 cards.
- There are $C_{64}^2 = \binom{64}{2} = \frac{64!}{2!62!} = 2\,016$ ways to place 2 black rooks on a chess board.

e) Multisets (or combinations with repetition)

Example

Let's draw 6 balls (with replacement i.e. repetition) out of an urn that contains 4 balls of 4 different colors: red, blue, green and yellow without considering the order. The result of this experiment is a **multiset**, i.e. a set in which the same element may be repeated several times, for example if we draw 1 red, 3 blue and 2 green balls we denote this by the multiset $\{R,B,B,B,G,G\}$. The number of these multisets or **combinations with repetition of 4 objects taken 6 at a time** is denoted by CR_4^6. To get this number the trick consists in adopting following notation:

$$\underbrace{\cdots\cdots\cdots}_{x\ \text{red balls}}\ |\ \underbrace{\cdots\cdots\cdots}_{y\ \text{blue balls}}\ |\ \underbrace{\cdots\cdots\cdots}_{z\ \text{green balls}}\ |\ \underbrace{\cdots\cdots\cdots}_{t\ \text{yellow balls}}$$

which contains $x+y+z+t=6$ "•" and 3 "|" to separate the 4 colors. Examples:

$\{R,B,B,B,G,G\}$ will be denoted: •|•••|••|

$\{R,R,G,Y,Y,Y\}$ will be denoted: ••||•|•••

$\{R,R,R,R,R,R\}$ will be denoted: ••••••|||, etc

The number of such sequences equals the number of ways you can chose 6 places among 10 for the "•" or 3 places among 10 for the "|" $C_9^6 = C_9^3 = 84$, so $CR_4^6 = C_{10}^6$.

General case

A c**ombination with repetition of n objects taken r at a time** or a draw of r objects without order and with repetition of an urn containing n objects is a multiset of r elements taken from a set with n different objects. The number of these combinations, denoted by CR_n^r or $\left(\!\!\binom{n}{p}\!\!\right)$, is given by the formula:

$$\boxed{CR_n^r = \left(\!\!\binom{n}{r}\!\!\right) = C_{n-1+r}^r}$$

Examples

- The number of solutions $(x;y;z)$ in nonnegative integers of the equation $x+y+z=8$ is $CR_3^8 = C_{10}^8 = 45$.
- When you buy an ice-cream with 4 scoops and you can choose among 15 flavors, (chocolate, strawberry,…), then there are $CR_{15}^4 = C_{18}^4 = 3060$ different ice-creams you can order.

Exercises 14 - 42

EXERCISES

1) Using Venn diagrams, analyse if:

a) Intersection, union and difference are associative laws.

b) Intersection is distributive over union and vice-versa.

c) Intersection is distributive over difference and vice-versa.

d) Union is distributive over difference and vice-versa.

2) Use Venn diagrams to state which of the following relations are correct for any sets A, B and C:

a) $A \cup (B \setminus C) = (A \cup B) \setminus C$

b) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

c) $(A \cup B) \setminus A = B$

d) $A \cup B \cup C = A \cup (B \setminus (A \cap B)) \cup (C \setminus (A \cap C))$

e) $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (A \cup B \cup C) \setminus (A \cap B \cap C)$

3) Let be $E = \{1;2;6;7;8;10;12\}$, $F = \{4;5;6;8;9\}$ and $G = \{2;3;4;8;10;11\}$.

a) Draw a Venn diagram.

b) Find expressions of the following sets using: E, F, G, ( ), $\cup$, $\cap$ and \:

- $\{8\}$
- $\{5;9\}$
- $\{4;5;6;9\}$
- $\{2;3;4;5;6;8;9;10;11\}$
- $\{2;4;6;8;10\}$
- $\{3;5;9;11\}$
- $\{2;4;6;10\}$
- $\{3;4;8;11\}$
- $\{2;5;9;10\}$

4) In a class of 30 students 9 like mathematics, 17 like biology, 13 like sports, 5 like mathematics and biology, 8 like biology and sports, 4 like math and sports and 3 like mathematics, biology and sports. How many students in this class don't like any of these subjects?

5) Among a group of 80 persons, some speak English, 40 persons speak French, some speak German and 10 of them don't speak any of these three languages. 20 persons speak 2 languages, 43 just one language, 41 do not speak English and 22 among them speak French and finally 18 persons speak only English.

a) How many members of this group speak the three languages?

b) Among the members of the group who speak two of these languages, how many speak French? German?

c) Among the members of the group who speak just one of these languages, how many speak English? French? German?

6) Let A, B, C, D be finite sets. Complete the following identities:

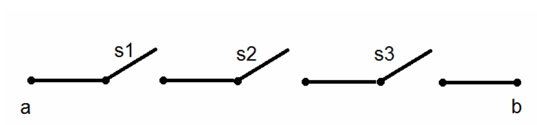
a) $|A \cup B| = |A| + |B| \ldots$

b) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| \ldots$

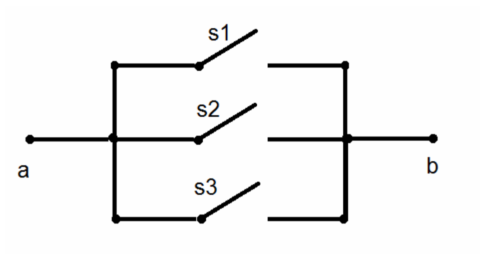
c) $|A \cup B \cup C \cup D| = \ldots$

7) Consider the switching networks shown in figure below. Let A_n denote the event that the switch s_n is closed, for $n = 1, 2, 3$. Let *A* denote the event that there is a closed path between terminals *a* and *b*. Using set operations, express *A* in terms of A_1, A_2, and A_3, for each of the networks shown.

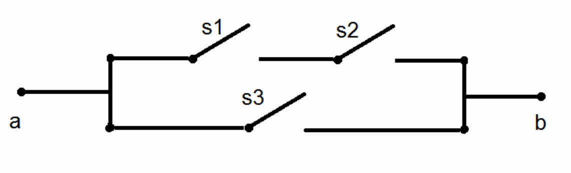
a)



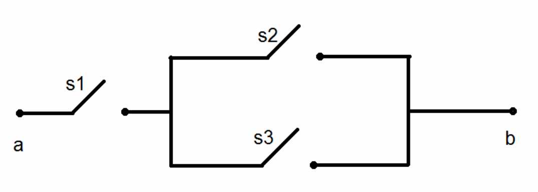
b)



c)



d)



8) Let A, B and C be arbitrary events of a sample space Ω. Find expressions (using A, B, C, ( ), $\cap$, $\cup$, $\overline{\ldots}$) for the events that, of A, B, C:

a) Only *A* occurs.

b) Both *A* and *B*, but not *C*, occur.

c) All three events occur.

d) At least one event occurs.

e) One event and no more occurs.

f) At least two events occur.

g) Two events and no more occur.

h) Not more than two events occur.

i) None of the three events occur.

9) Consider the experiment of drawing two cards at random from a bag containing four cards marked with the integers 1 through 4. Find the sample space Ω (and its cardinality) of the experiment

a) if the first card is replaced before the second is drawn

b) if the first card is not replaced before the second is drawn

c) if the two cards are drawn simultaneously so that order doesn't matter.

Is it reasonable to assume that all the sample points of Ω have the same chance to occur?

10) Find the sample space Ω and its cardinality in the following random experiences:

a) Place 3 undistinguishable (identical) balls into 4 distinguishable cells (numbered from 1 to 4 for example), each cell containing at most 1 ball.

b) Same as in a) but now each cell may contain at most 2 balls.

c) Same as in a) but now each cell may contain 3 balls.

Is it reasonable to assume that all the sample points of Ω have the same chance to occur?

11) Consider the experiment of drawing 3 balls at random from an urn containing 3 green balls and 2 red balls. We suppose that balls of the same color are undistinguishable. Find the sample space Ω of the experiment

a) if the balls are drawn one by one *with replacement* and considering the order in which they are drawn.

b) if the balls are drawn one by one *without replacement* and considering the order in which they are drawn.

c) if the 3 balls are drawn simultaneously.

Is it reasonable to assume that all the sample points of Ω have the same chance to occur?

12) Find the sample space Ω of the experiment consisting in rolling a die at most 4 times, stopping when a 6 appears. What is the number of sample points in Ω? Is it reasonable to assume that all the sample points of Ω have the same chance to occur? Determine the following events and the number of outcomes in each of them:

A: "get a 6 on the second roll"

B: "get a 6 on the third roll"

$C = \overline{A \cup B}$

13) Find the sample space Ω of the experiment consisting in rolling a die three times. What is the number of sample points in Ω? Is it reasonable to assume that all the sample points of Ω have the same chance to occur? For the following events try (if possible) to write them by using set operations and then calculate their probability:

A: "get no 6"

B: "get exactly one 6"

C: "the 6 appears at least twice"

D: "the product of the results is odd"

E: "the product of the results is even"

F: "get no 6 and no 1"

G: “6 or 1 appears at least once”

H: “6 and 1 appear both at least once”

I: “get at least one 6, but no 1”

14) From 6 men and 5 women, find the number of groups of 4 that can be formed consisting of 2 men and 2 women.

15) In a group of 20 people, if everyone shakes hands with everyone else, how many handshakes take place?

16) How many ways are there to put 2 black and 2 white rocks on a chessboard?

17) How many ways can a 3-people committee be chosen from 10 sets of identical twins such that no twins can both exist in this committee?

18) You have a box with sweets of 4 different colors: green, white, red and yellow. In how many ways can you choose 5 sweets from this box?

19) In an examination there are 6 multiple choice questions and each question has 4 choices. In how many ways a student can fail to get all answers correct?

20) How many anagrams (letter arrangements) can be formed using the letters of

a) the word “SALT”?

b) the word “CHILI”?

c) the word “PEPPER”?

21) How many ways can 5 boys and 3 girls be seated in a row if :

a) the 3 girls are to seat next to each other?

b) the 3 girls are not to seat next to each other.

22) Mr. Jones has 8 mathematics books, 3 chemistry books, 4 physics books and 5 history books. He put his books on the bookshelf without taking care of the order, so what’s the probability (specify the sample set Ω !) that

a) the books dealing with mathematics are not mixed up with books dealing with other subjects?

b) all the books dealing with the same subject are together on the shelf?

23) How many different linear arrangements are there of the letters A,B,C,D,E,F for which

a) A and B are next to each other?

b) A is before B?

c) A is before B and B is before C?

d) A is before B and C is before D?

e) A and B are next to each other and C and D are also next to each other ? f) E is not last in line?

24) In how many ways 5 boys and 5 girls can be seated at a round table if

a) no restriction is imposed?

b) each girl is to be seated between two boys?

c) Mary and Louise must sit together?

d) John and Edwin must not sit together?

e) all girls must sit together ?

25) How many ways are there to complete the following 3x3 sub-grids of a sudoku

(A)

	6	

(B)

		3
		1
		6

a) when no other constraints are known?

b) when the first row cannot contain the 2?

c) when the first row cannot contain the 2 and the middle column must contain the 4?

d) when the upper left cell can only contain 2 or 4?

26) If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many, if each school must receive at least 1 blackboard?

27) A dance class consists of 22 students, 10 women and 12 men. If 5 men and 5 women are to be chosen and then paired off, how many pairs are possible?

28) An urn contains 6 black balls, 5 white balls and 9 red balls. If you draw three balls (without replacement) out of this urn, what's the probability that you get:

a) one ball of each colour?

b) 3 balls of the same colour?

c) at least 2 black balls?

d) balls of 2 different colours?

29) A common pack of 52 playing cards includes 13 **ranks** (2, 3, …, 10, **jack** or **knave**, **queen**, **king**, **ace**) of each of the four **suits**, **clubs** (♣), **diamonds** (♦), **hearts** (♥) and **spades** (♠). A poker player gets a 5-card hand, in which order is not relevant. What's the probability (specify the sample set Ω !) that he gets:

a) a **straight flush** (i.e. 5 cards in sequence, all of the same suit, such as **Q♣ J♣ 10♣ 9♣ 8♣**)?

b) **four of a kind** (i.e. the 4 cards of one rank and any other unmatched card such as **9♣ 9♠ 9♦ 9♥ K♦**)?

c) **three of a kind** (i.e. 3 cards of the same rank, plus two cards which are not of this rank nor the same as each other such as **2♦ 2♠ 2♣ K♠ 6♥**)?

d) a **full house** (i.e. 3 matching cards of one rank and 2 matching cards of another rank such as **3♣ 3♠ 3♦ 6♣ 6♥**)?

e) a **flush** (i.e. all five cards are of the same suit, but not in sequence such as **Q♣ 10♣ 7♣ 6♣ 4♣**)?

f) a **straight** (i.e. 5 cards of sequential rank in at least two different suits such as **Q♣ J♠ 10♠ 9♥ 8♥**)?

g) a **two pair** hand (i.e. 2 cards of the same rank, plus 2 cards of another rank plus any card not of either rank such as **J♥ J♣ 4♣ 4♠ 9♥**)?

h) a **one pair** hand (i.e. 2 cards of the same rank, plus three cards which are not of this rank nor the same as each other such as **4♥ 4♠ K♠ 10♦ 5♠**)?

30) A player gets a hand of 6 cards out of a pack of 52. Calculate the probability (specify the sample set Ω !) that he gets:

a) 5 diamonds and 2 aces.

b) exactly one queen and 3 hearts.

c) exactly 2 kings, 2 tens and 2 clubs.

d) at least 2 clubs.

e) at least one card of every suit.

f) only 2 different ranks.

g) at the most 5 spades and at least 1 diamond.

31) An 8-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

32) The game of bridge is played by 4 players, each of whom is dealt 13 cards.

a) How many bridge deals are possible?

b) What is the probability that one of the players receives all 13 spades?

c) What is the probability that each player receives 1 ace?

33) Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.

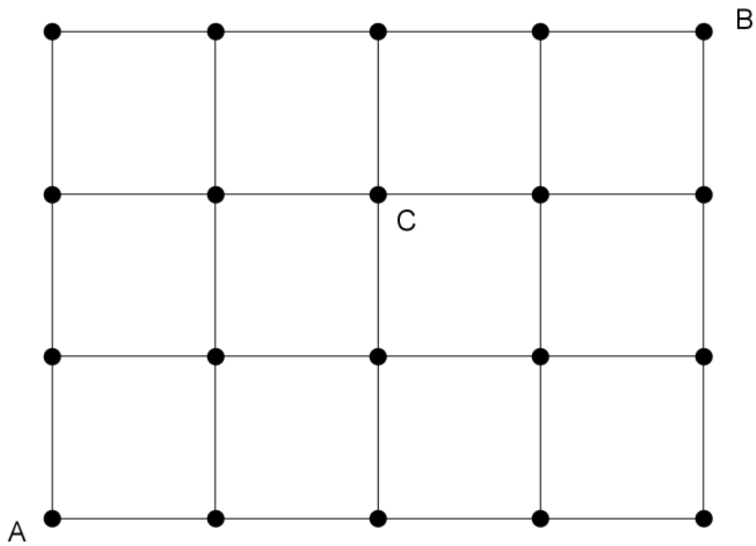
a) How many choices are there in which the 3 students are in the same class?

b) How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?

c) How many choices are there in which the 3 students are in different classes?

d) Using the results of parts a) through c), find a combinatorial identity!

34) Consider the 5×4 grid of points shown below. Suppose that starting at the point labeled A you can go one step up or one step to the right at each move.



a) How many different paths are possible from A to B?

b) How many different paths are possible from A to B that go through C?

c) How many different paths are possible from A to B that go not through C?

d) *A generalization:* Answer the question a) in a $p \times n$ grid.

35) If 8 rooks are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is, compute the probability that no row or file contains more than one rook.

36) Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

37) A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

38) How many positive integers not exceeding 2001 are multiples of 3 or 4, but not of 5?

39) Calculate the cardinality of the following sets:

$$A = \{(a;b;c;d) / a,b,c,d \in \mathbb{N} \text{ and } a+b+c+d=50\}$$

$$B = \{(a;b;c;d) / a,b,c,d \in \mathbb{N}^* \text{ and } a+b+c+d=50\}$$

40) A **partition** of a set S is a set of non-empty subsets of S such that the union of these subsets equals S and that these subsets are pair wise disjoint. In other words: every element of S is in exactly one of these subsets. For example if $S = \{a;b;c;d;e;f\}$ then $P_1 = \{S\}$ is the trivial partition of S in 1 subset (S itself), $P_2 = \{\{a;b;c;f\};\{d;e\}\}$ is a partition of S in 2 subsets and $P_3 = \{\{a\};\{b;c\};\{d;e;f\}\}$ a partition of S in 3 subsets. The number of partitions of a set of cardinality n is called the **Bell number** and denoted by B_n. Calculate the five first Bell numbers i.e. B_1, B_2, B_3, B_4, B_5.

41) In how many different ways can one select two distinct subsets of $S = \{a;b;c;d;e;f\}$ so that the union of the two subsets is S? (The order of selection does not matter; for example the pair of subsets $\{a;c\}$, $\{b;c;d;e;f\}$, represents the same selection as the pair $\{b;c;d;e;f\}$, $\{a;c\}$ and the two subsets are <u>not necessarily</u> disjoint as in a partition!)

42) Four cars enter a roundabout at the same time, each coming from a different direction, as show in the diagram. Each of the cars drives less than once round the roundabout and no two cars leave the roundabout in the same direction. How many different ways are there for the cars to leave the roundabout?