Question 1

$6iz^2 - 5(1+2i)z + 17 = 0$

$\Delta = 25(1+2i)^2 - 4\cdot 6\cdot 17i$
$= 25(1+4i-4) - 408i$
$= -75 - 308i$

rac. carrées de Δ : $\pm\delta = \pm(a+bi)$

$\begin{cases} a^2+b^2 = \sqrt{75^2+308^2} = 317 \\ a^2-b^2 = -75 \\ 2ab = -308 \text{ [signes diff.]} \end{cases}$ $\quad 2a^2 = 242$, $a^2 = 121$, $a = \pm 11$; $2b^2 = 392$, $b^2 = 196$, $b = \pm 14$

$\pm\delta = \pm(11-14i)$

solutions : $z_1 = \dfrac{5+10i+11-14i}{12i} = \dfrac{16-4i}{12i} = \dfrac{4-i}{3i} = \dfrac{-1-4i}{3}$

$z_2 = \dfrac{5+10i-11+14i}{12i} = \dfrac{-6+24i}{12i} = \dfrac{-1+4i}{2i} = \dfrac{4+i}{2}$

$S = \left\{ \dfrac{-1-4i}{3} ; \dfrac{4+i}{2} \right\}$

Question 2

a)

$z_1 = 2\operatorname{cis}\dfrac{3\pi}{4}$; $z_2 = 2\sqrt{6}\operatorname{cis}\left(-\dfrac{\pi}{6}\right)$

$\dfrac{z_1^5}{z_2^2} = \dfrac{2^5\operatorname{cis}\frac{15\pi}{4}}{24\operatorname{cis}\left(-\frac{\pi}{3}\right)} = \dfrac{4}{3}\operatorname{cis}\left(\dfrac{15\pi}{4}+\dfrac{\pi}{3}\right) = \dfrac{4}{3}\operatorname{cis}\dfrac{49\pi}{12} = \dfrac{4}{3}\operatorname{cis}\dfrac{\pi}{12}$

b) D'après a) : $z_1^5 = 2^5\operatorname{cis}\dfrac{15\pi}{4} = 32\operatorname{cis}\left(-\dfrac{\pi}{4}\right) = 32\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right) = 16\sqrt{2}(1-i)$,

et $z_2^2 = 24\operatorname{cis}\left(-\dfrac{\pi}{3}\right) = 24\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = 12(1-\sqrt{3}i)$

Par conséquent :

$$\begin{aligned}\frac{z_1^5}{z_2^2} &= \frac{16\sqrt{2}(1-i)}{12(1-\sqrt{3}i)} \\ &= \frac{4\sqrt{2}(1-i)(1+\sqrt{3}i)}{3(1-\sqrt{3}i)(1+\sqrt{3}i)} \\ &= \frac{4\sqrt{2}\left[(1+\sqrt{3})+(\sqrt{3}-1)i\right]}{3\cdot 4} \\ &= \frac{(\sqrt{6}+\sqrt{2})+(\sqrt{6}-\sqrt{2})i}{3}\end{aligned}$$

c)

$$\cos\frac{\pi}{12} = \frac{(\sqrt{6}+\sqrt{2})\cdot 3}{3\cdot 4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin\frac{\pi}{12} = \frac{(\sqrt{6}-\sqrt{2})\cdot 3}{3\cdot 4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Question 3

On cherche d'abord la forme algébrique de z :

$$\begin{aligned}
z &= \frac{\left(-22i-6\sqrt{3}\right)\left(5-2\sqrt{3}\,i\right)}{\left(5+2\sqrt{3}\,i\right)\left(5-2\sqrt{3}\,i\right)} \\
&= \frac{-110i-44\sqrt{3}-30\sqrt{3}+36i}{25+12} \\
&= \frac{-74\sqrt{3}-74i}{37} \\
&= -2\sqrt{3}-2i
\end{aligned}$$

On cherche ensuite la forme trigonométrique de z :

$$z = -2\sqrt{3}-2i = 4\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = 4\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)$$

Les racines 4^{es} de z sont donc :

$$\begin{aligned}
z_k &= \sqrt[4]{4}\,\mathrm{cis}\left(-\frac{5\pi}{24}+k\frac{2\pi}{4}\right) \\
&= \sqrt{2}\,\mathrm{cis}\left(-\frac{5\pi}{24}+k\frac{\pi}{2}\right),\quad k\in\{0,1,2,3\}
\end{aligned}$$

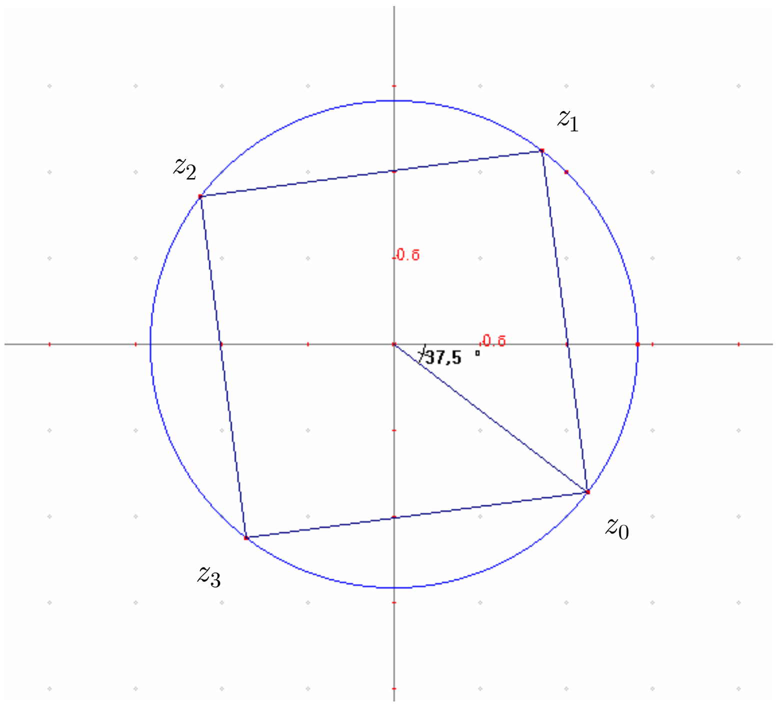
Plus précisément :

$$z_0 = \sqrt{2}\,\mathrm{cis}\left(-\frac{5\pi}{24}\right)$$

$$z_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{7\pi}{24}\right)$$

$$z_2 = \sqrt{2}\,\mathrm{cis}\left(\frac{19\pi}{24}\right)$$

$$z_3 = \sqrt{2}\,\mathrm{cis}\left(\frac{31\pi}{24}\right)$$



Question 4 (corrigé partiel !)

I) $\boxed{\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{e^x - e^{-x}}{2} = \mathrm{sh}(x) \end{array}}$ Remarquer que f est impaire !

1) • $D_f = \mathbb{R} = D_{f'}$, donc pas d'A.V.

• $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$ donc pas d'A.H.

• $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x \to \pm\infty} \dfrac{1}{2}\left(\dfrac{e^x}{x} - \dfrac{e^{-x}}{x}\right) = \lim\limits_{x \to \pm\infty} \dfrac{1}{2}\left(\dfrac{e^x}{x} - \dfrac{1}{xe^x}\right) = \left.\begin{array}{l} \text{"}+\infty - 0\text{"} \\ \text{si } x \to +\infty \\ \text{si } x \to -\infty \\ \text{"}0 - (-\infty)\text{"} \end{array}\right\} = +\infty$

car l'exponentielle l'emporte sur x

donc $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = +\infty$, donc pas d'A.O., (Bp de dir. Oy.)

2) $f'(x) = \dfrac{e^x + e^{-x}}{2}$ $(= \mathrm{ch}\, x)$, $\forall x \in \mathbb{R}$

x	$-\infty$	$+\infty$
$f'(x)$	+	
$f(x)$	$-\infty$ ↗	$+\infty$

3) $f''(x) = \dfrac{e^x - e^{-x}}{2} = f(x)$ $(\forall x \in \mathbb{R})$

$f''(x) = 0 \Leftrightarrow e^x = e^{-x}$ $\quad | \cdot e^x$

$\Leftrightarrow e^{2x} = 1$

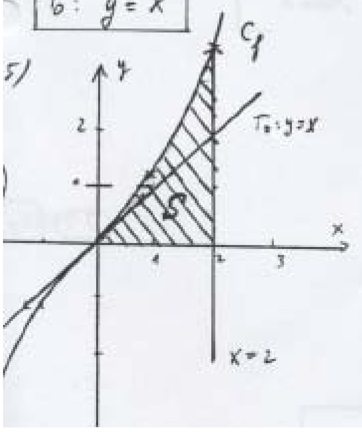
$\Leftrightarrow x = 0$ et comme

$f''(x)$ change de signe pour $x = 0$, l'origine est un point d'inflexion pour C_f

4) $T : y = f(x_0) + f'(x_0)(x - x_0)$, $x_0 = 0$ or $f(0) = 0$ et $f'(0) = 1$

$\boxed{T_0 : y = x}$

5)



$C_f \cap Ox = \{O(0,0)\}$

Question 6

2) $\dfrac{\ln x - 1}{\ln x - 2} < 3$ $(*)$

CE : 1) $x > 0$

2) $\ln x \neq 2 \Leftrightarrow x \neq e^2$

$D = \mathbb{R}_0^+ \setminus \{e^2\}$

$(*) \Leftrightarrow \dfrac{\ln x - 1}{\ln x - 2} - 3 < 0$

$\Leftrightarrow \dfrac{\ln x - 1 - 3\ln x + 6}{\ln x - 2} < 0$

$\Leftrightarrow \dfrac{-2\ln x + 5}{\ln x - 2} < 0$

Signe de $-2\ln x + 5$:

$-2\ln x + 5 = 0 \Rightarrow \ln x = \frac{5}{2} \Rightarrow x = e^{\frac{5}{2}}$

$-2\ln x + 5 > 0 \Rightarrow \ln x < \frac{5}{2} \Rightarrow x < e^{\frac{5}{2}}$

$-2\ln x + 5 < 0 \Rightarrow \ln x > \frac{5}{2} \Rightarrow x > e^{\frac{5}{2}}$

Signe de $\ln x - 2$:

$\ln x - 2 = 0 \Rightarrow x = e^2$

$\ln x - 2 > 0 \Rightarrow x > e^2$

$\ln x - 2 < 0 \Rightarrow x < e^2$

x	0		e^2		$e^{\frac{5}{2}}$		$+\infty$
$-2\ln x + 5$		+		+	0	−	
$\ln x - 2$		−	0	+		+	
$\dfrac{-2\ln x + 5}{\ln x - 2}$		−	||	+	0	−	

$S =]0, e^2[\cup]e^{\frac{5}{2}}, +\infty[$

Question 5

3) • $\displaystyle\lim_{x\to 0_+} x^2 \ln x^4 = \text{"}0\cdot(-\infty)\text{"}$ f.i.

$\displaystyle= \lim_{x\to 0_+} \frac{-\ln x^4}{\left(\frac{1}{x^2}\right)} = \text{"}\frac{-\infty}{+\infty}\text{"}$ f.i.

$\displaystyle\overset{(H)}{=} \lim_{x\to 0_+} \frac{\left(\frac{4}{x}\right)}{\left(\frac{-2}{x^3}\right)}$

$\displaystyle= \lim_{x\to 0_+} (-2x^2) = \boxed{0}$

• $\displaystyle\lim_{x\to 0_-} e^{\frac{1}{x}} \ln(x^2+1) = 0\cdot 0 = 0$

• $\displaystyle\lim_{x\to +\infty} \left(1+\frac{4}{x}\right)^{3x-7}$

$\displaystyle= \lim_{x\to +\infty} \left(1+\frac{1}{\left(\frac{x}{4}\right)}\right)^{3x} \cdot \left(1+\frac{4}{x}\right)^{-7}$

$\displaystyle= \lim_{\frac{x}{4}\to +\infty} \left[\left(1+\frac{1}{\left(\frac{x}{4}\right)}\right)^{\frac{x}{4}}\right]^{12} \underbrace{\left(1+\frac{4}{x}\right)^{-7}}_{\to 1}$

$= \boxed{e^{12}}$

G. Lorang