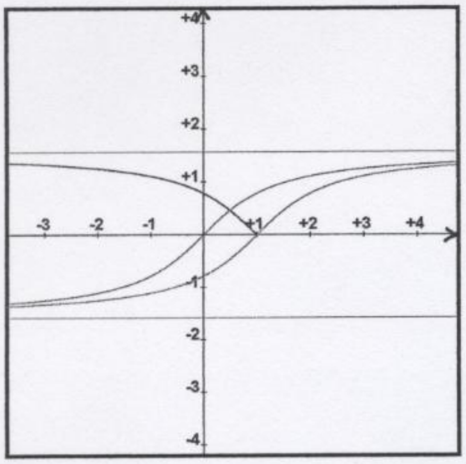
C et D Mathématiques II Session 2004 F1

Question I

a) cf. page 29. b) cf. page 31



c)

$$\boxed{y = \operatorname{Arctan} x}$$
↓ translation horizontale d'une unité vers la droite
$$\boxed{y = \operatorname{Arctan}(x-1)}$$
↓ • identité pour les pts. d'ord. positive
•• symétrie d'axe x pour les pts. d'ord. négative
$$\boxed{y = |\operatorname{Arctan}(x-1)|}$$

d) $\underline{f(x) = 2\cdot \operatorname{Arctan}\sqrt{e^x-1}}$

① CE : $e^x - 1 \geq 0 \Leftrightarrow e^x \geq 1 \Leftrightarrow x \geq 0$ | $\operatorname{dom} f = \mathbb{R}^+ = [0, +\infty[$; $\operatorname{dom} f' = \mathbb{R}^+_0 =]0, +\infty[$

② $\forall x > 0,\ f'(x) = \cancel{2}\cdot \dfrac{1}{1+(\sqrt{e^x-1})^2}\cdot\dfrac{e^x}{\cancel{2}\sqrt{e^x-1}} = \dfrac{e^x}{(\cancel{1}+e^x-\cancel{1})\sqrt{e^x-1}} = \dfrac{\cancel{e^x}}{\cancel{e^x}\sqrt{e^x-1}} = \underline{\underline{\dfrac{1}{\sqrt{e^x-1}}}}$

$$\begin{cases} f(\ln 2) = 2\cdot\operatorname{Arctan}\sqrt{e^{\ln 2}-1} = 2\cdot\operatorname{Arctan} 1 = 2\cdot\frac{\pi}{4} = \frac{\pi}{2} \\ f'(\ln 2) = \dfrac{1}{\sqrt{e^{\ln 2}-1}} = \dfrac{1}{\sqrt{1}} = 1 \end{cases}$$

$\left\ulcorner \lim_{x\to 0} f'(x) = +\infty \right\lrcorner$

$T_{\ln 2} \equiv y - \frac{\pi}{2} = 1\cdot(x - \ln 2)$

$\boxed{y = x + \frac{\pi}{2} - \ln 2} = x + 0{,}877\ldots$

Question II

a) $\boxed{2^{x+1} + 5\cdot 2^{-x} = 7}$ $D_E = \underline{\underline{\mathbb{R}}}$

$\Leftrightarrow 2\cdot 2^x + \frac{5}{2^x} - 7 = 0 \quad |\cdot 2^x > 0$

$\Leftrightarrow 2\cdot(2^x)^2 - 7\cdot 2^x + 5 = 0$

$\Leftrightarrow 2y^2 - 7y + 5 = 0$

| Pour $y = 2^x > 0$
| $\Delta = 49 - 40 = 9 > 0$

$\Leftrightarrow y = \frac{7+3}{2\cdot 2} = \frac{10}{4} = \frac{5}{2} > 0$ ✓ ou $y = \frac{7-3}{4} = \frac{4}{4} = 1 > 0$ ✓

Sol : 1) $2^x = \frac{5}{2}$ $\Updownarrow \log_2$ $x = \log_2(\frac{5}{2})$ | 2) $2^x = 1$ $\Leftrightarrow x = 0$

$$\underline{\underline{S = \{0 ; \log_2(\tfrac{5}{2})\}}}$$

b) $\boxed{\log_{\frac{1}{2}}(x^2-6x+5) \leq \log_2\left(\frac{1}{5}\right)}$

CE : $x^2-6x+5>0$ $\quad \Delta=16>0$

x	$-\infty$		1		5		$+\infty$
x^2-6x+5		$+$	0	$-$	0	$+$	

$D_E =]-\infty,1[\,\cup\,]5,+\infty[$

$$\Leftrightarrow \frac{\ln(x^2-6x+5)}{-\ln 2} \leq \frac{-\ln 5}{\ln 2} \quad \Big|\cdot(-\ln 2)<0$$

$$\Leftrightarrow \ln(x^2-6x+5) \geqslant \ln 5 \qquad (\ln \nearrow)$$

$$\Leftrightarrow x^2-6x+5 \geqslant 5$$

$$\Leftrightarrow x^2-6x \geqslant 0$$

$$\Leftrightarrow x(x-6) \geqslant 0$$

$$\Leftrightarrow x \in]-\infty,0]\cup[6,+\infty[= S_i$$

x	$-\infty$		0		6		$+\infty$
x^2-6x		$+$	0	$-$	0	$+$	

Sol : $S = S_i \cap D_E =]-\infty,0]\cup[6,+\infty[$

c) $\boxed{f(x) = \frac{1}{x}\ln\frac{1+x}{1-x}}$

① CE : $\begin{cases} x\neq 0 & (1) \\ x\neq 1 & \\ \frac{1+x}{1-x}>0 & (2) \end{cases}$

x		-1			1	
$1+x$	$-$	0	$+$			$+$
$1-x$	$+$		$+$		0	$-$
$\frac{1+x}{1-x}$	$-$	0	$+$		‖	$-$

$\mathrm{dom}\,f =]-1;0[\,\cup\,]0;1[= \mathrm{dom}\,f'$

Parité de f :

$\forall x \in \mathrm{dom}\,f$, $f(-x) = \frac{1}{-x}\cdot\ln\frac{1-x}{1+x} = -\frac{1}{x}\cdot\left(-\ln\frac{1+x}{1-x}\right) = \frac{1}{x}\cdot\ln\frac{1+x}{1-x} = f(x)$

$\Rightarrow f$ est paire $\qquad * \ \ln\frac{a}{b} = -\ln\frac{b}{a}$

② Limites aux bornes de dom f.

$\cdot \ \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1^-} \underbrace{\frac{1}{x}}_{\to 1}\ln\frac{1+x \ (\to 2)}{1-x \ (\to 0^+)} \ (\to +\infty) = \boxed{+\infty}$

$\cdot\cdot$ f étant paire : $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to(-1)^+} f(x) = \boxed{+\infty}$

$\therefore \ \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\frac{1}{x}\ln\frac{1+x}{1-x}$ fi $[0\cdot\infty]$

$$= \lim_{x\to 0}\frac{\ln\frac{1+x}{1-x}}{x} \quad \left[\frac{0}{0}\right] \text{ fi}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{\frac{2}{1-x^2}}{1} \qquad \left(\ln\frac{1+x}{1-x}\right)' = \frac{1}{\frac{1+x}{\cancel{1-x}}}\cdot\frac{\overbrace{1\cdot(1-x)-(1+x)\cdot(-1)}^{2}}{(1-x)^{\cancel{2}}}$$

$$= \lim_{x\to 0}\frac{2}{1-x^2} = \boxed{2}$$

③ $\lim\limits_{x\to 0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\frac{1}{x}\cdot\ln\frac{1+x}{1-x}} \overset{②}{=} \boxed{e^2}$

fi $[1^\infty]$

$$\boxed{f(x) = \frac{\ln x}{\sqrt{x}}}$$

a) CE : $x > 0$ $\quad$ $\text{dom} f =]0, +\infty[$

Signe de $f(x)$: $\quad f(x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$ $\quad$ Signe de $f(x)$ = signe $\ln x$

x	0		1		$+\infty$
$f(x)$	//	$-$	0	$+$	

$\ln x > 0 \Leftrightarrow x > 1$

b) • $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{\ln x \to -\infty}{\sqrt{x} \to 0^+} = \boxed{-\infty} \Rightarrow \underline{AV \equiv \boxed{x = 0}}$

•• $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{\ln x}{\sqrt{x}}$ $\quad \left[\dfrac{\infty}{\infty}\right]$ f.i.

$$\overset{(H)}{=} \lim_{x \to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to +\infty} \frac{2\sqrt{x}}{x} = \lim_{x \to +\infty} \frac{2}{\sqrt{x}} = \boxed{0}$$

$\Rightarrow \underline{AH_d \equiv \boxed{y = 0}}$

c) • $\text{dom} f' =]0, +\infty[$

•• $\forall x > 0, \quad f'(x) = \dfrac{\frac{1}{x} \cdot \sqrt{x} - \frac{1}{2\sqrt{x}} \cdot \ln x}{(\sqrt{x})^2} = \dfrac{\frac{2 \cdot 1}{2 \cdot \sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x} = \boxed{\dfrac{2 - \ln x}{\underbrace{2x\sqrt{x}}_{>0}}}$

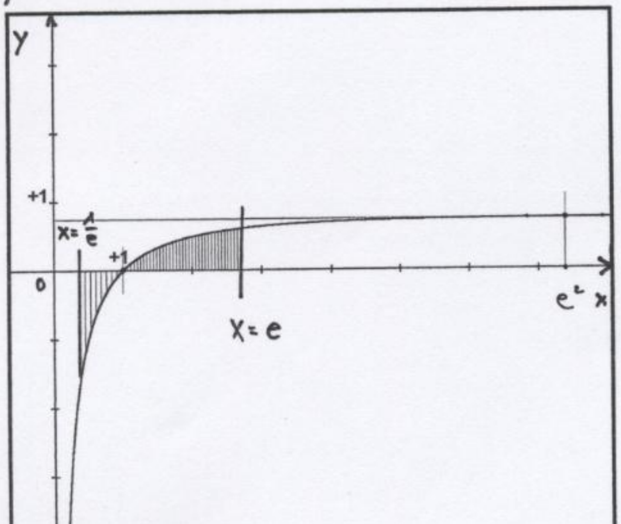
Signe de $f'(x)$ = signe de $(2 - \ln x)$

$f'(x) = 0 \Leftrightarrow 2 - \ln x = 0 \Leftrightarrow \ln x = 2 \Leftrightarrow x = e^2$ $\quad$ et $\quad f(e^2) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$

Tableau des variations :

x	0		e^2		$+\infty$
$f'(x)$	//	$+$	0	$-$	
$f(x)$	//	$-\infty \nearrow$	$\frac{2}{e}$	$\searrow$	0

d)



e) ① Aire A

$$A = -\int_{\frac{1}{e}}^{1} f(x)\,dx + \int_{1}^{e} f(x)\,dx = F(e) + F\left(\tfrac{1}{e}\right) - 2F(1)$$

$F(x) = \int x^{-\frac{1}{2}} \cdot \ln x \cdot dx$ $\quad$ IPP $\quad \left\| \begin{array}{ll} u(x) = \ln x & v'(x) = x^{-\frac{1}{2}} \\ u'(x) = \frac{1}{x} & v(x) = 2\sqrt{x} \end{array} \right.$

$= 2\sqrt{x} \ln x - 2 \cdot \int x^{-\frac{1}{2}}\,dx$

$= 2\sqrt{x} \ln x - 4\sqrt{x} + \cancel{k}$

$= \underline{2\sqrt{x}\,(\ln x - 2)}$

$F(e) = 2\sqrt{e}\,(1 - 2) = -2\sqrt{e}$ $\quad \Big| \quad F(1) = 2\sqrt{1}\,(0 - 2) = -4$

$F\left(\frac{1}{e}\right) = \frac{2}{\sqrt{e}}\,(-1 - 2) = -\frac{6}{\sqrt{e}}$

$$A = -2\sqrt{e} - \frac{6}{\sqrt{e}} + 8 = 8 - \frac{2e + 6}{\sqrt{e}} = \underline{\underline{\frac{8\sqrt{e} - 2e - 6}{\sqrt{e}}}} \ \underset{\text{cm}^2}{\text{ua}} \simeq \underline{1{,}06}337\ldots \text{ cm}^2$$

② <u>Volume V</u>

$$V = \pi \cdot \int_{\frac{1}{e}}^{e} [f(x)]^2 dx = \pi \cdot \int_{\frac{1}{e}}^{e} \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx = \pi \int_{\frac{1}{e}}^{e} \frac{(\ln x)^2}{x} dx = \pi \cdot \left[F(e) - F\left(\frac{1}{e}\right)\right]$$

$$F(x) = \int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \cdot \frac{1}{x} \cdot dx$$ <u>Poser</u> : $t = \ln x$, $dt = \frac{1}{x} dx$

$$= \int t^2 dt = \frac{t^3}{3} + k = \underline{\underline{\frac{(\ln x)^3}{3} + k}}$$

Soit $\underline{F(x) = \frac{(\ln x)^3}{3}}$: $F(e) = \frac{(1)^3}{3} = \frac{1}{3}$; $F\left(\frac{1}{e}\right) = \frac{(-1)^3}{3} = -\frac{1}{3}$ $\Big\}$ $V = \pi \cdot \left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] = \pi \cdot \frac{2}{3} = \boxed{\frac{2\pi}{3}}$ uv cm³

<u>Question IV</u>

$$\boxed{f(x) = \text{Arc sin } \frac{e^{2x}-1}{e^{2x}+1}}$$

a) CE : $\overbrace{-1 \leq \frac{e^{2x}-1}{e^{2x}+1}}^{(I)}$, $\underbrace{\frac{e^{2x}-1}{e^{2x}+1} \leq 1}_{(II)}$

(I) $\frac{e^{2x}-1}{e^{2x}+1} + 1 \geqslant 0 \Leftrightarrow \frac{\overbrace{2e^{2x}}^{>0}}{\underbrace{e^{2x}+1}_{>0}} \geqslant 0$ tj vrai : $\forall x \in \mathbb{R}$, $\frac{2e^{2x}}{e^{2x}+1} > 0$

(II) $\frac{e^{2x}-1}{e^{2x}+1} - 1 \leq 0 \Leftrightarrow \frac{\overset{<0}{-2}}{\underbrace{e^{2x}+1}_{>0}} \leq 0$ tj vrai : $\forall x \in \mathbb{R}$, $\frac{-2}{e^{2x}+1} < 0$

$$\text{dom} f = \text{dom} f' = \boxed{\mathbb{R}}$$

b) $\forall x \in \mathbb{R}$, $f'(x) = \frac{1}{\sqrt{1 - \left(\frac{e^{2x}-1}{e^{2x}+1}\right)^2}} \cdot \frac{\overbrace{2e^{2x}(e^{2x}+1) - 2e^{2x}(e^{2x}-1)}^{4e^{2x}}}{(e^{2x}+1)^2}$

$$= \frac{4e^{2x}}{\sqrt{\frac{(e^{2x}+1)^2 - (e^{2x}-1)^2}{(e^{2x}+1)^2}} \cdot (e^{2x}+1)^2} = \frac{4e^{2x}}{\frac{\sqrt{4e^{2x}}}{\cancel{e^{2x}+1}} \cdot (e^{2x}+1)^{\cancel{2}}} = \frac{\overset{2}{\cancel{4}}e^{\cancel{2x}}}{\underset{1}{\cancel{2}}\cancel{e^x}(e^{2x}+1)} = \boxed{\frac{2e^x}{e^{2x}+1}}$$

c) • $I = \int_0^{\frac{1}{2}\ln 3} \frac{e^x}{e^{2x}+1} dx \overset{b)}{=} \frac{1}{2} \text{Arcsin } \frac{e^{2x}-1}{e^{2x}+1} \Big]_0^{\ln\sqrt{3}} = \frac{1}{2}\left[\text{Arcsin} \overset{= \frac{1}{2}}{\left(\frac{3-1}{3+1}\right)} - \text{Arcsin } 0\right]$

$$= \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\pi}{12}}$$

<u>Variante</u> : $I = \left[\text{Arctan } e^x\right]_0^{\ln\sqrt{3}} = \text{Arctan} \sqrt{3} - \text{Arctan } 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

•• $\int \frac{e^{2x}}{e^{2x}+1} dx = \frac{1}{2} \cdot \int \frac{2e^{2x}}{e^{2x}+1} dx = \frac{1}{2} \cdot \int \frac{1}{t} dt$ <u>Poser</u> $t = e^{2x}+1 > 0$, $dt = 2e^{2x} dx$

$$= \frac{1}{2} \ln |t| + k = \underline{\underline{\frac{1}{2} \ln (e^{2x}+1) + k}}$$

$$J = \int_{-1}^{1} \frac{e^{2x}}{e^{2x}+1} dx = \left[\frac{1}{2} \ln (e^{2x}+1)\right]_{-1}^{1} = \frac{1}{2}\left[\ln(e^2+1) - \ln \frac{e^2+1}{e^2}\right] = \frac{1}{2} \ln \frac{\cancel{(e^2+1)} \cdot e^2}{\cancel{e^2+1}} = \frac{1}{2} \cdot 2 = \boxed{1}$$