Corrigé modèle

I) a) voir livre p. 29

1 b) $\lim\limits_{x\to 0} \dfrac{\text{Arctan } 3x}{2x} \quad \left(= \dfrac{0}{0} \text{ f.i.}\right)$

2 $\overset{H}{=} \lim\limits_{x\to 0} \dfrac{\frac{3}{1+(3x)^2}}{2}$

$= \dfrac{3}{2}$

c) $f(x) = 3\,\text{Arctan}\,x - \pi$

1. $\text{dom} f = \mathbb{R}$

Comme Arctan est str. ↗ sur $\mathbb{R}$, f est str ↗ sur $\mathbb{R}$

1,5 $\lim\limits_{x\to -\infty} f(x) = 3\cdot\left(-\dfrac{\pi}{2}\right) - \pi = -\dfrac{5\pi}{2}$

$\lim\limits_{x\to +\infty} f(x) = 3\cdot\dfrac{\pi}{2} - \pi = \dfrac{\pi}{2}$

$\text{im} f = \left]\lim\limits_{-\infty} f(x) ; \lim\limits_{+\infty} f(x)\right[= \left]-\dfrac{5\pi}{2} ; \dfrac{\pi}{2}\right[$

10 2. Soit $x \in \text{dom} f$: $y = f(x) \qquad y \in \left]-\dfrac{5\pi}{2} ; \dfrac{\pi}{2}\right[$

$y = 3\,\text{Arctan}\,x - \pi$

2 $y + \pi = 3\,\text{Arctan}\,x$

$\dfrac{y+\pi}{3} = \text{Arctan}\,x$

$\tan\dfrac{y+\pi}{3} = x$

D'où : $f^{-1}(x) = \tan\dfrac{x+\pi}{3}$

3. $\text{dom} f^{-1} = \left]-\dfrac{5\pi}{3}, \dfrac{\pi}{2}\right[$

0,5 $\text{im} f^{-1} = \mathbb{R}$

4. Sur $[-1;1]$ f est négative.

Aire : $-\displaystyle\int_{-1}^{1} f(x)\,dx = -\int_{-1}^{1} (3\,\text{Arctan}\,x - \pi)\,dx$

3 $= -3\displaystyle\int_{-1}^{1} \text{Arctan}\,x\,dx + \int_{-1}^{1} \pi\,dx$

- Arctan est impaire donc $\displaystyle\int_{-a}^{a} \text{Arctan}\,x = 0$
- ou ipp.

$= -3\cdot 0 + [\pi\cdot x]_{-1}^{1}$

$= \pi - (-\pi)$

$= 2\pi$

II) $\begin{cases} f(x) = \text{Arctan}\,\dfrac{1}{(x-2)^2} \\ f(2) = \dfrac{\pi}{2} \end{cases}$ ②

1. dom $f = \mathbb{R}$

1 $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \text{Arctan}\,\underbrace{\dfrac{1}{(x-2)^2}}_{\to +\infty}$

$= 0$

2. · $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \text{Arctan}\,\underbrace{\dfrac{1}{(x-2)^2}}_{\to 0^+} \Big\} \to +\infty$

$= \dfrac{\pi}{2}$

1 $= f(2)$ Donc f est continue en 2.

· $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2} \dfrac{\text{Arctan}\,\frac{1}{(x-2)^2} - \frac{\pi}{2}}{x-2}$ $\left(= \dfrac{0}{0}\ \text{f.i.}\right)$

$\overset{H}{=} \lim\limits_{x\to 2} \dfrac{\frac{-2}{(x-2)^3}}{1+\left(\frac{1}{(x-2)^2}\right)^2}$ $\left(\dfrac{1}{(x-2)^2}\right)' = \dfrac{-2}{(x-2)^3}$

2 $= \lim\limits_{x\to 2} \dfrac{-2 \cdot (x-2)^4}{[(x-2)^4+1](x-2)^3}$

10 $= \lim\limits_{x\to 2} \dfrac{-2\cdot(x-2)}{(x-2)^4+1}$

$= 0$ Donc f est dérivable en 2.

3. Pour $x \neq 2$, $f'(x) = \dfrac{-2\cdot(x-2)}{\underbrace{(x-2)^4+1}_{>0}}$

Donc f' a le signe contraire de $x-2$.

Tableau de variation :

2

x	$-\infty$		2		$+\infty$
$f'(x)$		+	0	−	
$f(x)$	0	↗	$\frac{\pi}{2}$	↘	0

4. Tangente au point d'abscisse 1 :

$y = f'(a)\cdot(x-a) + f(a)$

$a = 1$

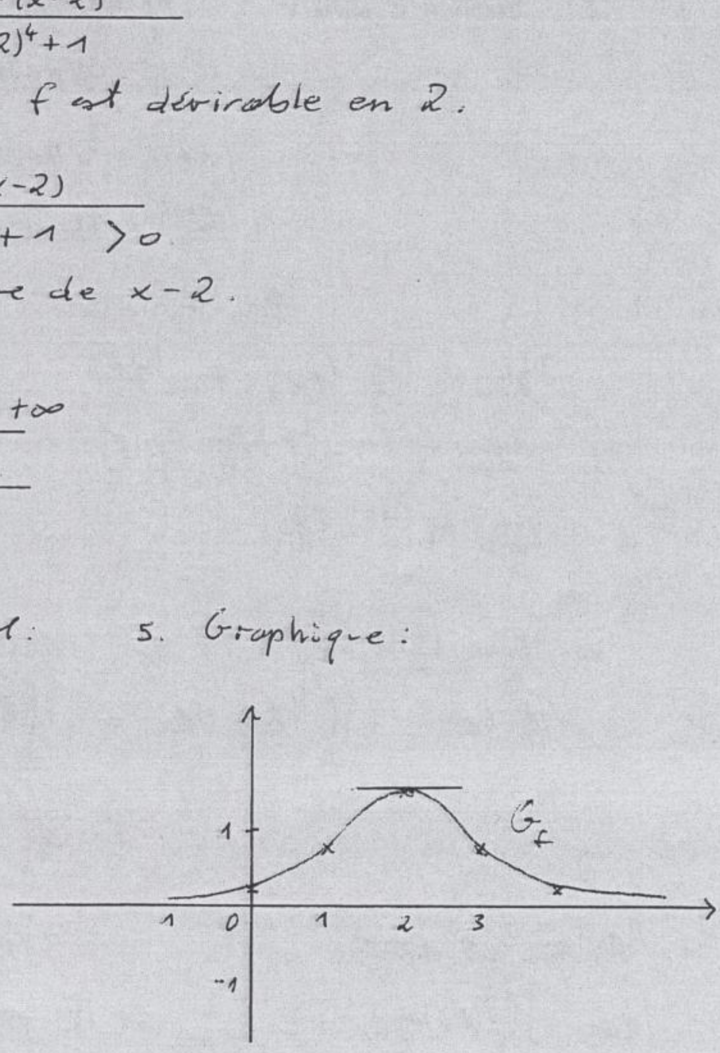
2 $f(1) = \text{Arctan}\,1 = \dfrac{\pi}{4}$

$f'(1) = \dfrac{-2\cdot(-1)}{1+1} = 1$

D'où : $y = x - 1 + \dfrac{\pi}{4}$

5. Graphique :

2



x	−1	0	1	3	4
$f(x)$	0,11	0,24	0,78	0,78	0,24

III) a) voir livre p.56

1 b) 1. $f(x) = 3^{\text{Arcsin}x}$ $\qquad f'(x) = 3^{\text{Arcsin}x} \cdot \ln 3 \cdot (\text{Arcsin}x)'$

Cond.: $-1 \leq x \leq 1$ $\qquad = \dfrac{3^{\text{Arcsin}x} \cdot \ln 3}{\sqrt{1-x^2}}$

2 dom $f = [-1; 1]$

dom$_d$ $f =]-1; 1[$

2. $f(x) = [\log_5(x^2+5)]^3$ $\qquad f'(x) = 3\,[\log_5(x^2+5)]^2 \cdot \dfrac{1}{\ln 5} \cdot \dfrac{2x}{x^2+5}$

2 Cond.: $\underbrace{x^2}_{\geq 0}+5 > 0$ $\qquad = \dfrac{6x \cdot [\log_5(x^2+5)]^2}{(x^2+5) \cdot \ln 5}$

dom $f = \mathbb{R}$ = dom$_d$ f

c) 1. $6 \cdot 25^x - 22 \cdot 5^x = 40$ $\qquad D = \mathbb{R}$

$6 \cdot 5^{2x} - 22 \cdot 5^x - 40 = 0$

Posons $y = 5^x$: $\qquad 6y^2 - 22y - 40 = 0$

$\Delta = 484 + 960 = 1444$

2,5 $y = \dfrac{22+38}{12} = 5$ ou $y = \dfrac{22-38}{12} = -\dfrac{16}{12} = -\dfrac{4}{3}$

D'où : $5^x = 5$ ou $5^x = -\frac{4}{3}$

11 à rejeter

$x = 1$

$S = \{1\}$

2. $\begin{cases} \log_9 y = \frac{1}{2} + \frac{1}{2}\log_3 x \\ xy = 48 \end{cases}$ $\qquad$ or $\log_9 y = \dfrac{\log_3 y}{\log_3 9} = \dfrac{1}{2}\log_3 y$

Cond.: $x > 0$ et $y > 0$!

$\begin{cases} \log_3 y = \log_3 3 + \log_3 x \\ xy = 48 \end{cases}$

3,5 $\begin{cases} y = 3x \\ 3x^2 = 48 \end{cases}$

$\begin{cases} y = 3x = 12 \\ x = 4 \text{ ou } x = -4 \end{cases}$

à rejeter

$S = \{(4; 12)\}$

IV) a) 1. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = \int -\frac{2\cos x\cdot(-\sin x)}{1+\cos^2 x}\,dx$ Forme : $\dfrac{u'}{1+u}$ (4)

$= -\ln(1+\cos^2 x) + k \quad k\in\mathbb{R}$

1,5

2. $\displaystyle\int \cos 3x\cdot\cos x\,dx = \int \frac{1}{2}(\cos 4x + \cos 2x)\,dx$

$= \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + k \quad k\in\mathbb{R}$

b) 1. $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}\cdot \operatorname{Arccos} x}\,dx = \left[-\ln(\operatorname{Arccos} x)\right]_0^{\frac{\sqrt{3}}{2}}$

$= -\ln\frac{\pi}{6} + \ln\frac{\pi}{2}$

2

$= \ln\dfrac{\frac{\pi}{2}}{\frac{\pi}{6}}$

$= \ln 3$

2. $\displaystyle\int_{\sqrt{\frac{3}{2}}}^{\sqrt{3}} \frac{x}{\sqrt{9-x^4}}\,dx$ or $\dfrac{x}{\sqrt{9-x^2}} = \dfrac{x}{3\cdot\sqrt{1-\left(\frac{x^2}{3}\right)^2}}$

$= \dfrac{1}{2}\cdot\dfrac{\frac{2x}{3}}{\sqrt{1-\left(\frac{x^2}{3}\right)^2}}$

11

$= \left[\frac{1}{2}\cdot \operatorname{Arcsin}\frac{x^2}{3}\right]_{\sqrt{\frac{3}{2}}}^{\sqrt{3}}$

$= \frac{1}{2}\operatorname{Arcsin} 1 - \frac{1}{2}\operatorname{Arcsin}\frac{1}{2}$

2

$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{6}$

$= \frac{\pi}{4} - \frac{\pi}{12}$

$= \frac{3\pi-\pi}{12}$

$= \frac{\pi}{6}$

3. $\displaystyle\int_0^2 \sqrt{16-x^2}\,dx = \int_0^2 4\sqrt{1-\left(\frac{x}{4}\right)^2}\,dx$ Posons $\frac{x}{4} = \cos t$

$= \displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4\sqrt{1-\cos^2 t}\cdot(-4)\sin t\,dt$ alors $x = 4\cdot\cos t = g(t)$

et $g'(t) = -4\cdot\sin t$

$= \displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -16\cdot\underbrace{|\sin t|}_{\geq 0 \text{ sur } [\frac{\pi}{6};\frac{\pi}{2}]}\cdot\sin t\,dt$ $x=0 \Rightarrow t=\frac{\pi}{2}$

$x=2 \Rightarrow t=\frac{\pi}{3}$

$= -16\cdot\displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sin^2 t\,dt$

4

$= -16\displaystyle\int\left(\frac{1}{2}-\frac{1}{2}\cos 2t\right)dt$

$= -16\cdot\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$

$= -16\cdot\left[\left(\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right)-\left(\frac{\pi}{4}-0\right)\right]$

$= -\frac{8}{3}\pi + 2\sqrt{3} + 4\pi = \frac{4\pi}{3} + 2\sqrt{3}$

V) $f(x) = (x+1)\, e^{\frac{1}{x+1}}$ (5)

1. $\mathrm{dom}\, f = \mathbb{R} - \{-1\}$

0,5

- $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \underbrace{(x+1)}_{\to\pm\infty} \cdot e^{\overbrace{\frac{1}{x+1}}^{\to 0}}$

1 $\qquad = \pm\infty$ pas d'A.H.

- A.O. ? $\lim\limits_{\pm\infty} \dfrac{f(x)}{x} = \lim\limits_{\pm\infty} (1 + \underbrace{\tfrac{1}{x}}_{\to 0}) \cdot e^{\underbrace{\frac{1}{x+1}}_{\to 0}}$

$= 1$

$\lim\limits_{\pm\infty} [f(x) - x] = \lim\limits_{\pm\infty} \left[(x+1)\cdot e^{\frac{1}{x+1}} - x\right]$

$= \lim\limits_{\pm\infty} \left(x \cdot e^{\frac{1}{1+x}} - x + e^{\frac{1}{x+1}}\right)$

3 $\qquad = \lim\limits_{\pm\infty} \left[x\cdot\left(e^{\frac{1}{1+x}} - 1\right) + e^{\frac{1}{x+1}}\right]$

$= 1 + 1$ $\qquad$ or $\lim\limits_{\pm\infty} x\cdot\left(e^{\frac{1}{1+x}} - 1\right) = \lim\limits_{\pm\infty} \dfrac{e^{\frac{1}{1+x}} - 1}{\frac{1}{x}}$

$= 2$ $\qquad \overset{H}{=} \lim\limits_{\pm\infty} \dfrac{e^{\frac{1}{1+x}} \cdot \left(\frac{-1}{(1+x)^2}\right)}{-\frac{1}{x^2}}$

(12) A.O. $y = x + 2$! $\qquad = \lim\limits_{\pm\infty} \underbrace{e^{\frac{1}{1+x}}}_{\to 1} \cdot \underbrace{\dfrac{x^2}{(1+x)^2}}_{\to 1}$

$= 1$

- $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \underbrace{(x+1)}_{\to 0} \cdot e^{\overbrace{\frac{1}{x+1}}^{\to -\infty}}$

0,5 $\qquad = 0$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \underbrace{(x+1)}_{\to 0} \cdot e^{\overbrace{\frac{1}{x+1}}^{\to +\infty}}$

$\left(= \lim\limits_{x\to -1^+} \dfrac{x+1}{e^{-\frac{1}{x+1}}} \quad \left(\tfrac{0}{0} \text{ f.i.}\right) \right.$

2 $\quad \overset{H}{=} \lim\limits_{x\to -1^+} \dfrac{1}{e^{-\frac{1}{x+1}} \cdot \frac{1}{(x+1)^2}}$

$= \lim\limits_{x\to -1^+} \dfrac{(x+1)^2}{e^{-\frac{1}{x+1}}}$

$\left. = \tfrac{0}{0} \text{ f.i.} \right)$

$= \lim\limits_{x\to -1^+} \dfrac{e^{\frac{1}{x+1}}}{\frac{1}{x+1}} \quad \left(\tfrac{\infty}{\infty} \text{ f.i.}\right.$

$\overset{H}{=} \lim\limits_{x\to -1^+} \dfrac{e^{\frac{1}{1+x}} \cdot \frac{-1}{(1+x)^2}}{\frac{-1}{(1+x)^2}}$

$= +\infty$

A.V. $x = 1$

2. $f'(x) = 1 \cdot e^{\frac{1}{x+1}} + (x+1)\cdot e^{\frac{1}{x+1}} \cdot \dfrac{-1}{(x+1)^2}$

$= e^{\frac{1}{x+1}} \cdot \left(1 - \dfrac{1}{x+1}\right)$

$= \dfrac{x}{x+1} \cdot \underbrace{e^{\frac{1}{x+1}}}_{>0}$

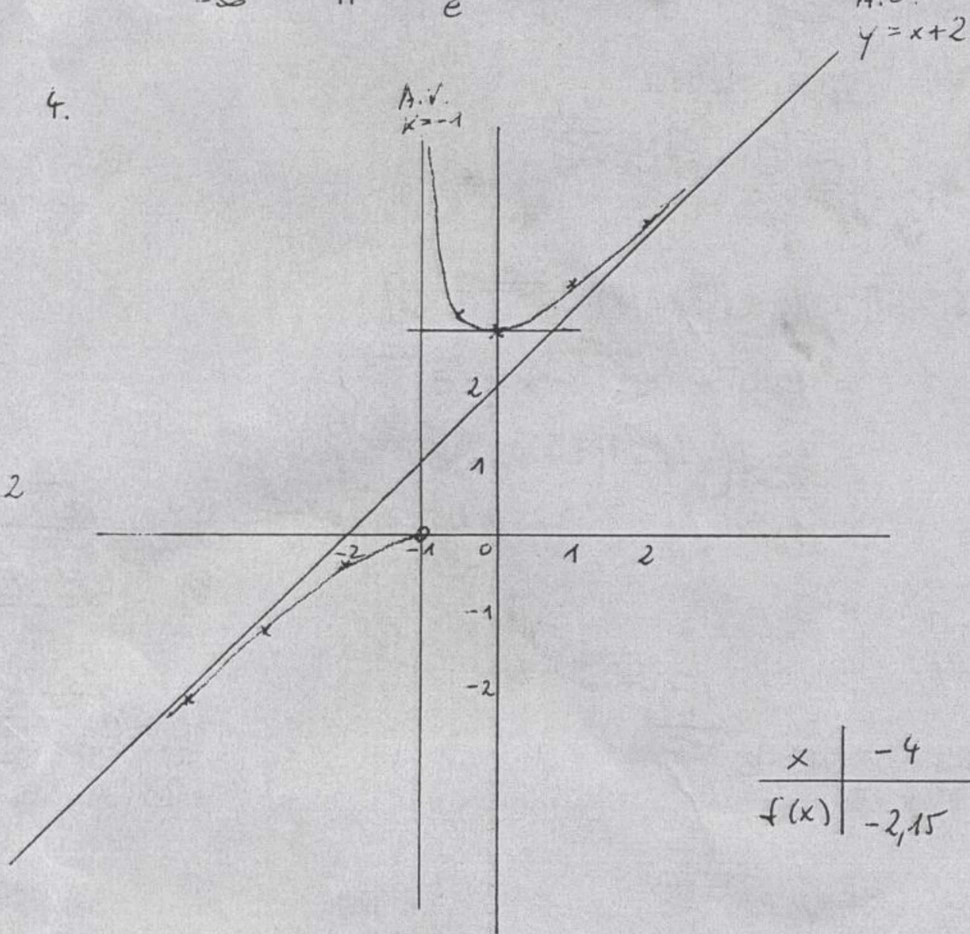
Signe de $\dfrac{x}{x+1}$:

x		-1		0	
x	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
$\frac{x}{x+1}$	$+$	‖	$-$	0	$+$

Tableau de variation.

x	$-\infty$		-1		0		$+\infty$
$f'(x)$		$+$	||	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	0 || $+\infty$	$\searrow$	e	$\nearrow$	$+\infty$

4.



x	-4	-3	-2	1	2	$0{,}5$
$f(x)$	$-2{,}15$	$-1{,}21$	$-0{,}37$	$3{,}3$	$4{,}18$	$2{,}9$

$\overline{\text{VI}}$ 1. $f(x)$ a le même signe que x c.à.d. $f(x) < 0$ pour $x<0$
$f(x) \geq 0$ pour $x \geq 0$

0,5

2. Volume: $V(\lambda) = \pi \cdot \int_\lambda^0 [f(x)]^2 dx$

$= \pi \cdot \int_\lambda^0 x^2 e^x \, dx$ ipp $f'(x) = e^x$ $f(x) = e^x$, $g(x) = x^2$ $g'(x) = 2x$ 1

$= \pi \cdot \left\{ [x^2e^x]_\lambda^0 - \int_\lambda^0 2xe^x \, dx \right\}$ ipp $f'(x) = e^x$ $f(x) = e^x$, $g(x) = 2x$ $g'(x) = 2$ 1

$= \pi \cdot \left\{ 0 - \lambda^2 e^\lambda - \left([2xe^x]_\lambda^0 - \int_\lambda^0 2e^x dx \right) \right\}$

$= \pi \cdot \left\{ -\lambda^2 e^\lambda - (0 - 2\lambda e^\lambda) + [2e^x]_\lambda^0 \right\}$

$= \pi \cdot (-\lambda^2 e^\lambda + 2\lambda e^\lambda + 2 - 2e^\lambda)$ 1

Or $\lim\limits_{\lambda \to -\infty} \lambda e^\lambda = \lim\limits_{\lambda \to -\infty} \frac{\lambda}{e^{-\lambda}} \overset{H}{=} \lim\limits_{\lambda \to -\infty} \frac{1}{-e^{-\lambda}} = 0$

et $\lim\limits_{-\infty} \lambda^2 e^\lambda = \lim\limits_{-\infty} \frac{\lambda^2}{e^{-\lambda}} \overset{H}{=} \lim\limits_{-\infty} \frac{2\lambda}{-e^{-\lambda}} = 0$

D'où $\lim\limits_{-\infty} V(\lambda) = \pi \cdot 2 = 2\pi$. 0,5