Corrigé

I 1) a)

$$z = \frac{(\sqrt{3} + i + \sqrt{3}i - 1)(1 - i\sqrt{3})}{4} = \frac{\sqrt{3} + i + i\sqrt{3} - 1 - 3i + \sqrt{3} + 3 + i\sqrt{3}}{4}$$

$$z = \frac{2 + 2\sqrt{3} + (-2 + 2\sqrt{3})i}{4} = \boxed{\frac{1+\sqrt{3}}{2} + \frac{-1+\sqrt{3}}{2}i}$$

b)

$$z = \frac{[\sqrt{2}, \frac{\pi}{4}][2, \frac{\pi}{6}]}{[2, \frac{\pi}{3}]} = \left[\sqrt{2}, \frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{3}\right] = \left[\sqrt{2}, \frac{3\pi + 2\pi - 4\pi}{12}\right] = \left[\sqrt{2}, \frac{\pi}{12}\right]$$

$$z = \boxed{\sqrt{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)}$$

c) Soit $\delta = [r, \theta]$ avec $r > 0$ et $\theta \in \mathbb{R}$.

$$\delta^3 = z \iff [r^3, 3\theta] = \left[\sqrt{2}, \frac{\pi}{12}\right]$$

$$\iff \begin{cases} r^3 = \sqrt{2} = 2^{\frac{1}{2}} \\ 3\theta = \frac{\pi}{12} + k2\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} r = 2^{\frac{1}{6}} = \sqrt[6]{2} \\ \theta = \frac{\pi}{36} + k\frac{2\pi}{3}, \ k \in \mathbb{Z} \end{cases}$$

Racines cubiques de z : $\boxed{\left[\sqrt[6]{2}, \frac{\pi}{36}\right], \left[\sqrt[6]{2}, \frac{25\pi}{36}\right], \left[\sqrt[6]{2}, \frac{49\pi}{36}\right]}$

2)

$$\Delta = (5+7i)^2 - 4(1+i)(10+4i)$$
$$= 25 + 70i - 49 - 4(10 + 4i + 10i - 4) = -24 + 70i - 24 - 56i$$
$$= -48 + 14i$$

Soit $\delta = x + iy$ avec $x \in \mathbb{R}$ et $y \in \mathbb{R}$.

$$\delta^2 = -48 + 14i \iff \begin{cases} x^2 - y^2 = -48 & (1) \\ xy = 7 & (2) \\ x^2 + y^2 = \sqrt{48^2 + 14^2} = 50 & (3) \end{cases}$$

$(1) + (3)$: $2x^2 = 2$, $x^2 = 1$	$(3) - (1)$: $2y^2 = 98$, $y^2 = 49$	(2) x et y sont de même signe

Finalement : $\delta = 1 + 7i$ ou $\delta = -1 - 7i$

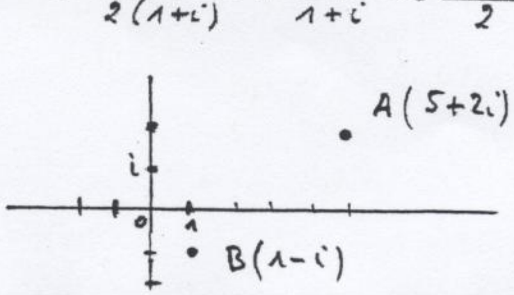
Solutions :

$$z_1 = \frac{5 + 7i + 1 + 7i}{2(1+i)} = \frac{6 + 14i}{2(1+i)} = \frac{3 + 7i}{1+i} = \frac{(3+7i)(1-i)}{2}$$

$$z_1 = \frac{3 - 3i + 7i + 7}{2} = 5 + 2i$$

$$z_2 = \frac{5 + 7i - 1 - 7i}{2(1+i)} = \frac{4}{2(1+i)} = \frac{2}{1+i} = \frac{2(1-i)}{2} = 1 - i$$

$$\boxed{S = \{5 + 2i, 1 - i\}}$$



3) $z_1 = \sqrt{2} + \sqrt{2}\, i$

$r_1 = \sqrt{2+2} = 2$

$\cos\alpha_1 = \frac{\sqrt{2}}{2}$

$\sin\alpha_1 = \frac{\sqrt{2}}{2}$

$z_1 = \left[2, \frac{\pi}{4}\right]$

$z_2 = \frac{\sqrt{3}}{\sqrt{5}} + \frac{1}{\sqrt{5}} i$

$r_2 = \sqrt{\frac{3}{5} + \frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

$\cos\alpha_2 = \frac{\frac{\sqrt{3}}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{3}}{2}$

$\sin\alpha_2 = \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{1}{2}$

$z_2 = \left[\frac{2}{\sqrt{5}}, \frac{\pi}{6}\right]$

a) $\frac{z_1}{z_2} = \frac{\left[2, \frac{\pi}{4}\right]}{\left[\frac{2}{\sqrt{5}}, \frac{\pi}{6}\right]} = \left[\frac{2}{\frac{2}{\sqrt{5}}}, \frac{\pi}{4} - \frac{\pi}{6}\right] = \left[\sqrt{5}, \frac{\pi}{12}\right]$

$$\boxed{\frac{z_1}{z_2} = \sqrt{5}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)}$$

b) $\frac{z_1}{z_2} = \frac{\sqrt{2} + \sqrt{2}\, i}{\frac{\sqrt{3} + i}{\sqrt{5}}} = \frac{\sqrt{5}\sqrt{2}(1+i)}{\sqrt{3} + i} = \frac{\sqrt{10}\,(1+i)(\sqrt{3} - i)}{4}$

$$\boxed{\frac{z_1}{z_2} = \frac{\sqrt{10}}{4}\left[(\sqrt{3}+1) + i(\sqrt{3}-1)\right]}$$

c) $\sqrt{5}\cos\frac{\pi}{12} = \frac{\sqrt{10}}{4}(\sqrt{3}+1)$

$\sqrt{5}\sin\frac{\pi}{12} = \frac{\sqrt{10}}{4}(\sqrt{3}-1)$

$$\boxed{\begin{aligned} \cos\frac{\pi}{12} &= \frac{\sqrt{2}}{4}(\sqrt{3}+1) = \frac{\sqrt{6}+\sqrt{2}}{4} \\ \sin\frac{\pi}{12} &= \frac{\sqrt{2}}{4}(\sqrt{3}-1) = \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}}$$

II 1) $A\cdot B = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & \alpha \end{pmatrix}$

$B\cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & \alpha \end{pmatrix}$

On a : $A\cdot B \neq B\cdot A$

2) $\begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & \alpha \end{vmatrix} = -\alpha + 8 - 6 + 2 - 12 + 2\alpha = \alpha - 8$

$\begin{matrix} 1 & -1 & 2 \\ 2 & -1 & 6 \end{matrix}$

3) A est singulière $\Leftrightarrow$ $\det A = 0 \Leftrightarrow \alpha - 8 = 0 \Leftrightarrow \alpha = 8$

4) $A^{-1} = \dfrac{1}{\det A}\ \mathrm{adj}.\ A$

$= \dfrac{1}{-8}\begin{pmatrix} -12 & 4 & -4 \\ 6 & -2 & -2 \\ 5 & -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{8} & \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$

III
$$\begin{cases} mx + y + z = 1 \\ x + z = 1 \\ x + y = 1 \end{cases} \quad (E)$$

$$\Delta = \begin{vmatrix} m & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \overset{C_1/C_1-C_2}{=} \begin{vmatrix} m-1 & 1 & 1 \\ 1 & 0 & 1 \\ \boxed{0 \quad 1 \quad 0} \end{vmatrix} = - \begin{vmatrix} m-1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -(m-1-1) = -(m-2) = 2-m \qquad \boxed{\Delta = 2-m}$$

$\boxed{1^{er}\text{ cas}}$ $\Delta = 0 \iff 2-m=0 \iff \underline{m=2}$

$$(E) \iff \begin{cases} 2x + y + z = 1 & (1) \\ x + z = 1 & (2) \\ x + y = 1 & (3) \end{cases} \iff \begin{cases} (2)\ z = 1-x \\ (1)\ 2x + y + 1 - x = 1 \\ (3)\ x + y = 1 \end{cases} \iff \begin{cases} z = 1-x \\ x+y=0 \\ x+y=1 \end{cases}$$

impossible

$\underline{S = \emptyset}$

Dans un repère de l'espace (E) est un système de trois plans n'ayant aucun point commun.

$\boxed{2^e\text{ cas}}$ $\Delta \neq 0 \iff \underline{m \neq 2}$

$$\Delta_x = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \overset{C_1/C_1-C_2}{=} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$\Delta_y = \begin{vmatrix} m & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \overset{C_3/C_3-C_2}{=} \begin{vmatrix} m & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1) \begin{vmatrix} m & 1 \\ 1 & 1 \end{vmatrix} = -(m-1) = 1-m$$

$$\Delta_z = \begin{vmatrix} m & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \overset{C_2/C_2-C_3}{=} \begin{vmatrix} m & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} m & 1 \\ 1 & 1 \end{vmatrix} = 1-m$$

$$x = \frac{\Delta x}{\Delta} = \frac{1}{2-m} \quad ; \quad y = \frac{\Delta y}{\Delta} = \frac{1-m}{2-m} \quad ; \quad z = \frac{\Delta z}{\Delta} = \frac{1-m}{2-m}$$

$$\underline{S = \left\{ \left(\frac{1}{2-m}, \frac{1-m}{2-m}, \frac{1-m}{2-m} \right) \right\}}$$

(E) est un système de trois plans sécants en un point.