Mathématiques II- CD - Corrigé modèle

Question 1 **(4+3=7pts)**

1) $$\lim_{x\to+\infty}\left(\frac{x-1}{x+2}\right)^{2x+1}=\lim_{x\to+\infty}\left(\frac{x+2-3}{x+2}\right)^{2x+1}=\lim_{x\to+\infty}\left(1+\frac{1}{-\frac{x+2}{3}}\right)^{-\left(\frac{x+2}{3}\right)\cdot(-6)-3}$$

$$=\lim_{x\to+\infty}\left[\underbrace{\left(1+\frac{1}{-\frac{x+2}{3}}\right)^{-\left(\frac{x+2}{3}\right)}}_{\to e}\right]^{-6}\cdot\underbrace{\left(1+\frac{1}{-\frac{x+2}{3}}\right)^{-3}}_{\to 1}=e^{-6}=\frac{1}{e^{6}}$$

2) $$\lim_{x\to 0}\frac{\log(1-x^2)}{\sin^2 x}=_H\lim_{x\to 0}\frac{\frac{1}{1-x^2}\cdot(-2x)}{2\cdot ln10\cdot\sin x\cdot\cos x}=\lim_{x\to 0}\frac{-2x}{ln10\cdot(1-x^2)\cdot\sin 2x}$$

$$=_H\lim_{x\to 0}\frac{-2}{-2x\cdot ln10\cdot\sin 2x+ln10\cdot(1-x^2)\cos 2x\cdot 2}=-\frac{2}{2ln10}=-\frac{1}{ln10}$$

Question 2 **(7+5=12pts)**

1) $\log_{\sqrt{2}}(5-x)+\log_{\frac{1}{2}}(2x^2+5x-3)\le 2$

$C.E.: 1)\ 5-x>0 \iff x<5$

$2)\ 2x^2+5x-3>0 \iff x\in\,]-\infty,-3[\,\cup\,\left]\frac{1}{2};+\infty\right[$

$\left[\Delta=49,\ x_1=-3,\ x_2=\frac{1}{2}\right]$

$D=]-\infty,-3[\,\cup\,\left]\frac{1}{2};5\right[$

$\log_{\sqrt{2}}(5-x)+\log_{\frac{1}{2}}(2x^2+5x-3)\le 2$

$\iff \frac{\ln(5-x)}{\ln 2^{\frac{1}{2}}}+\frac{\ln(2x^2+5x-3)}{\ln 2^{-1}}\le 2$

$\iff 2\ln(5-x)-\ln(2x^2+5x-3)\le 2\ln 2$

$\iff \ln(5-x)^2\le \ln 4+\ln(2x^2+5x-3)$

$\iff (5-x)^2\le 4(2x^2+5x-3)$

$\iff -7x^2-30x+37\le 0$

$\Delta=900+4\cdot 7\cdot 37=1936$

$x_1=\frac{30-44}{-14}=1 \qquad x_2=\frac{30+44}{-14}=-\frac{37}{7}$

x	$-\infty$ $\quad -\frac{37}{7} \quad$ 1 $\quad +\infty$
$-7x^2-30x+37$	$-\quad 0 \quad + \quad 0 \quad -$

$S = \left]-\infty; -\frac{37}{7}\right] \cup [1; 5[$

2) $\frac{7e^x-2e^{-x}}{3e^x-1} = 1 - e^{-x}$

$C.E.: 3e^x - 1 \neq 0 \Leftrightarrow e^x \neq \frac{1}{3} \Leftrightarrow x \neq -\ln 3$

$D = \mathbb{R} \setminus \{-\ln 3\}$

$\frac{7e^x - 2e^{-x}}{3e^x - 1} = 1 - e^{-x}$

$\Leftrightarrow 7e^x - 2e^{-x} = (1 - e^{-x})(3e^x - 1)$

$\Leftrightarrow 7e^x - 2e^{-x} = 3e^x - 1 - 3 + e^{-x}$

$\Leftrightarrow 4e^x - 3e^{-x} = -4$

$\Leftrightarrow 4e^{2x} - 3 = -4e^x$

$\Leftrightarrow 4e^{2x} + 4e^x - 3 = 0$

Posons: $y = e^x > 0$

Il faut donc résoudre: $4y^2 + 4y - 3 = 0$

$\Delta = 16 - 4 \cdot 4 \cdot (-3) = 64$

$y_1 = \frac{-4+8}{8} = \frac{1}{2} \qquad y_2 = \frac{-4-8}{8} = -\frac{3}{2}$

Il reste donc à résoudre:

$e^x = \frac{1}{2} \quad ou \quad \underbrace{e^x}_{>0} = -\frac{3}{2}\ impossible$

$\Leftrightarrow x = -\ln 2$

$S = \{-\ln 2\}$

Question 3 **(4+3+4+3+6=20pts)**

1) $D_f = \mathbb{R}$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \underbrace{e^{1-x}}_{\to 0}\left(\underbrace{-2x^2 - x - 1}_{\to -\infty}\right) = \lim\limits_{x\to+\infty} \frac{-2x^2 - x - 1}{e^{x-1}} =_H \lim\limits_{x\to+\infty} \frac{-4x-1}{e^{x-1}}$

$=_H \lim\limits_{x\to+\infty} -\frac{4}{\underbrace{e^{x-1}}_{\to+\infty}} = 0 \qquad A.H.D: y = 0$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \underbrace{e^{1-x}}_{\to+\infty}\Big(\underbrace{-2x^2 - x - 1}_{\to-\infty}\Big) = -\infty$$ pas d'$A.H.G.$

Cauchy:

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \underbrace{e^{1-x}}_{\to+\infty}\Big(\underbrace{-2x - 1 - \frac{1}{x}}_{\to+\infty}\Big) = +\infty$$ $B.P.G\ d'axe\ (Oy)$

2) $D_{f'} = \mathbb{R}$

$$f'(x) = -e^{1-x}(-2x^2 - x - 1) + e^{1-x}(-4x - 1) = \underbrace{e^{1-x}}_{>0}(2x^2 - 3x)$$

$$f'(x) = 0 \Leftrightarrow 2x^2 - 3x = 0 \Leftrightarrow x(2x-3) = 0 \Leftrightarrow x = 0\ ou\ x = \frac{3}{2}$$

x	-∞		0		$\frac{3}{2}$		+∞
$f'(x)$		+	0	-	0	+	
$f(x)$	$-\infty$	↗	max $-e$ $\approx -2,72$	↘	min $-\frac{7}{\sqrt{e}}$ $\approx -4,25$	↗	0

Maximum en $A(0;-e)$ et minimum en $B\left(\frac{3}{2}, -\frac{7}{\sqrt{e}}\right)$

3) $D_{f''} = \mathbb{R}$

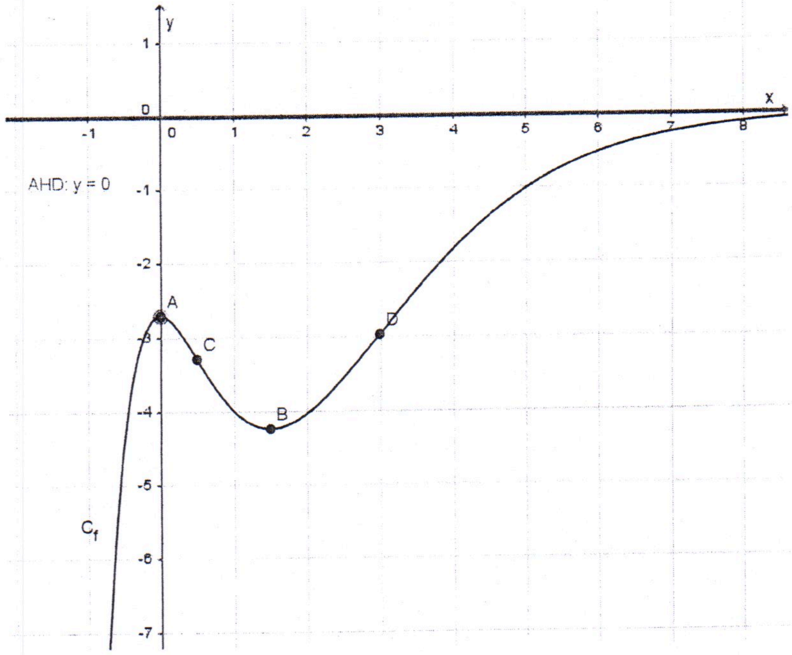
$$f''(x) = -e^{1-x}(2x^2 - 3x) + e^{1-x}(4x - 3) = e^{1-x}(-2x^2 + 7x - 3)$$

$$f''(x) = 0 \Leftrightarrow -2x^2 + 7x - 3 = 0 \Leftrightarrow x = \frac{1}{2}\ ou\ x = 3 \qquad \left[\Delta = 25,\ x_1 = \frac{1}{2},\ x_2 = 3\right]$$

x	-∞		$\frac{1}{2}$		3		+∞
$f''(x)$		-	0	+	0	-	
$f(x)$		∩	P.I. $-2\sqrt{e}$ ≈ -3.3	∪	P.I. $-22e^{-2}$ ≈ -3	∩	

Points d'inflexion en $C\left(\frac{1}{2}; -2\sqrt{e}\right)$ et en $D\left(3; -\frac{22}{e^2}\right)$

4)



5) $A_\lambda = -\int_0^\lambda e^{1-x}(-2x^2 - x - 1)dx = \int_0^\lambda e^{1-x}(2x^2 + x + 1)dx$

$IPP \quad f(x) = 2x^2 + x + 1 \qquad g'(x) = e^{1-x}$

$\qquad f'(x) = 4x + 1 \qquad g(x) = -e^{1-x}$

$$A_\lambda = [-e^{1-x}(2x^2 + x + 1)]_0^\lambda + \int_0^\lambda e^{1-x}(4x + 1)dx$$

$IPP\ f(x) = 4x + 1 \qquad g'(x) = e^{1-x}$

$\qquad f'(x) = 4 \qquad g(x) = -e^{1-x}$

$$A_\lambda = [-e^{1-x}(2x^2 + x + 1)]_0^\lambda + [-e^{1-x}(4x + 1)]_0^\lambda + \int_0^\lambda 4e^{1-x}\,dx$$

$$= [-e^{1-x}(2x^2 + 5x + 6)]_0^\lambda = -e^{1-\lambda}(2\lambda^2 + 5\lambda + 6) + 6e^1$$

$$\lim_{x\to+\infty} A_\lambda = \lim_{x\to+\infty}\left(-\underbrace{e^{1-\lambda}}_{\to 0}\left(\underbrace{2\lambda^2 + 5\lambda + 6}_{\to+\infty}\right) + 6e\right) =_H \lim_{x\to+\infty}\left(-\frac{4\lambda + 5}{e^{\lambda-1}} + 6e\right)$$

$$=_H \lim_{x\to+\infty}\left(\underbrace{-\frac{4}{e^{\lambda-1}}}_{\to 0} + 6e\right) = 6e$$

Question 4 (7+3=10pts)

1) $C.E.$: (1) $\frac{x+1}{2x-1} > 0$ (2) $2x - 1 \neq 0 \Leftrightarrow x \neq \frac{1}{2}$

Tableau des signes:

x	$-\infty$		-1		$\frac{1}{2}$		$+\infty$
$x+1$		-	0	+		+	
$2x-1$		-		-	0	+	
$\frac{x+1}{2x-1}$		+	0	-	||	+	

$D_f =]-\infty; -1[\cup \left]\frac{1}{2}; +\infty\right[$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(x - \ln\overbrace{\frac{x+1}{2x-1}}^{\to \ln\frac{1}{2}}\right) = \pm\infty$$ pas d'$A.H.$

On a: $\lim_{x\to\pm\infty} \frac{x+1}{2x-1} = \lim_{x\to\pm\infty} \frac{x}{2x} = \frac{1}{2}$

Recherche d'une $A.O.$ éventuelle:

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x - \ln\frac{x+1}{2x-1}}{x} = \lim_{x\to\pm\infty}\left(1 - \overbrace{\frac{1}{x}}^{\to 0} \cdot \ln\overbrace{\frac{x+1}{2x-1}}^{\to \ln\frac{1}{2}}\right) = 1\ (= a)$$

$$\lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty}\left[-\ln\overbrace{\frac{x+1}{2x-1}}^{\to \ln\frac{1}{2}}\right] = -\ \ln\frac{1}{2} = \ln 2\ \ (= b)$$

$A.O. \equiv y = x + \ln 2$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(\overbrace{x}^{\to -1} - \ln\overbrace{\frac{x+1}{2x-1}}^{\to 0^+}\right) = +\infty$$ $A.V: x = -1$

$$\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^+}\left(\overbrace{x}^{\to\frac{1}{2}} - \ln\overbrace{\frac{x+1}{2x-1}}^{\to+\infty}\right) = -\infty \qquad A.V.: x = \frac{1}{2}$$

2) $\forall x \in]-\infty; -1[\cup \left]\frac{1}{2}; +\infty\right[$:

$$\varphi(x) = f(x) - y_{A.O.} = -\ln\frac{x+1}{2x-1} - \ln 2 = -\left(\ln\frac{x+1}{2x-1} + \ln 2\right) = -\ln\frac{2x+2}{2x-1}$$

Résolvons:

$$\varphi(x) > 0 \Leftrightarrow -\ln\frac{2x+2}{2x-1} > 0 \Leftrightarrow \ln\frac{2x+2}{2x-1} < 0 \Leftrightarrow \frac{2x+2}{2x-1} < 1 \Leftrightarrow \frac{3}{2x-1} < 0 \Leftrightarrow x < \frac{1}{2}$$

x	$-\infty$		-1		$\frac{1}{2}$		$+\infty$
$\varphi(x)$		$+$	‖		‖	$-$	
Position		$C_f / A.O$	‖		‖	$A.O. / C_f$	

<u>Question 5</u> (3+3=6pts)

1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+\sin x)^2 dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1 + 2\sin x + \sin^2 x)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1 + 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cdot\frac{1}{2}\cdot\cos 2x \cdot 2\right)dx = \left[\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left[\frac{\pi}{2} - 1 - \frac{\sqrt{3}}{8}\right] - \left[\frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8}\right] = \frac{\pi}{2} - 1 - \frac{\sqrt{3}}{8} - \frac{\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{8} = \frac{\pi}{4} + \sqrt{3} - 1$$

2) $\int \frac{2-x}{\sqrt{1-9x^2}}dx = \int\left(\frac{2}{\sqrt{1-9x^2}} - \frac{x}{\sqrt{1-9x^2}}\right)dx$

$$= \int\left(\frac{2}{3}\cdot\frac{1}{\sqrt{1-(3x)^2}}\cdot 3 + \frac{1}{18}\cdot(-18x)\cdot(1-9x^2)^{-\frac{1}{2}}\right)dx$$

$$= \frac{2}{3}\cdot Arc\ \sin 3x + \frac{1}{18}\cdot\frac{1}{\frac{1}{2}}\cdot(1-9x^2)^{\frac{1}{2}} + c\ (c \in \mathbb{R})$$

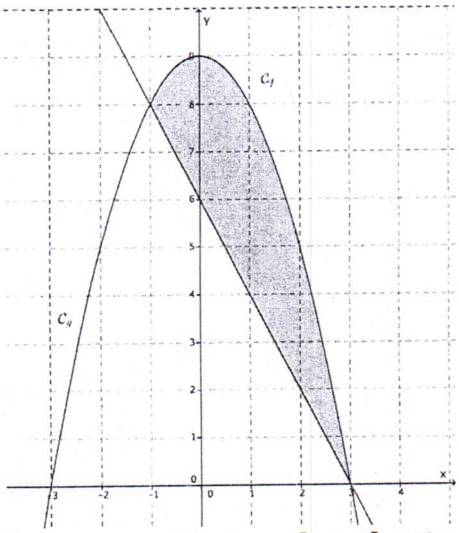
$$= \frac{2}{3}Arc\ \sin 3x + \frac{1}{9}\sqrt{1-9x^2} + c\ (c \in \mathbb{R})$$

<u>Question 6</u> (5pts)

$f(x) = 9 - x^2$et $g(x) = -2x + 6$

$f(x) = g(x) \Leftrightarrow\ 9 - x^2 = -2x + 6 \Leftrightarrow x^2 - 2x - 3 = 0 \Leftrightarrow x = -1\ ou\ x = 3$

$[\Delta = 16,\ x_1 = -1,\ x_2 = 3]$



f et g sont positives sur $[-1; 3]$ et $f \geq g$ sur $[-1; 3]$.

$$V = \pi \cdot \int_{-1}^{3} [f^2(x) - g^2(x)]\, dx = \pi \cdot \int_{-1}^{3} [(9 - x^2)^2 - (-2x + 6)^2]\, dx$$

$$= \pi \cdot \int_{-1}^{3} (81 - 18x^2 + x^4 - 4x^2 + 24x - 36)\, dx$$

$$= \pi \cdot \int_{-1}^{3} (x^4 - 22x^2 + 24x + 45)\, dx$$

$$= \pi \cdot \left[\frac{1}{5}x^5 - \frac{22}{3}x^3 + 12x^2 + 45x\right]_{-1}^{3}$$

$$= \pi \cdot \left[\left(\frac{243}{5} - 198 + 108 + 135\right) - \left(-\frac{1}{5} + \frac{22}{3} + 12 - 45\right)\right]$$

$$= \pi \cdot \left(\frac{468}{5} + \frac{388}{15}\right) = \frac{1792\pi}{15}\ cm^3 \approx\ 375{,}3\ cm^3$$
