

Corrigé

Question I (2 + 4 = 6 points)

- 1) voir manuel EM66 à la page 55
- 2) voir manuel EM66 à la page 87

Question II (4 + 7 = 11 points)

1) $8e^{-3x} - e^{3x} \geq -7 \mid \cdot e^{3x} > 0$

dom = \mathbb{R}

$\Leftrightarrow 8 - e^{6x} \geq -7e^{3x}$

$\Leftrightarrow -e^{6x} + 7e^{3x} + 8 \geq 0$ (1)

Posons $y = e^{3x} > 0$. (1) devient $-y^2 + 7y + 8 \geq 0$ $\Delta = 81$ $y_1 = -1$ et $y_2 = 8$

y	$-\infty$	-1	0	8	$+\infty$
$-y^2 + 7y + 8$	$-$	0	$+$	$+$	0

Ainsi $0 < y \leq 8$, donc $e^{3x} \leq 8$ et par conséquent $x \leq \frac{\ln 2^3}{3} = \ln 2$.

$S =]-\infty; \ln 2]$

2) $x + \log_2(2^x - 0,5) = \log_{0,5} 9$

CE: $2^x - 0,5 > 0$

$\Leftrightarrow \log_2 2^x + \log_2(2^x - 0,5) = \frac{\log_2 9}{\log_2 0,5}$

$\Leftrightarrow 2^x > 2^{-1}$

$\Leftrightarrow \log_2 2^x(2^x - 0,5) = -\log_2 9$

$\Leftrightarrow x > -1$

$\Leftrightarrow \log_2(2^{2x} - \frac{1}{2} \cdot 2^x) = \log_2 \frac{1}{9}$

dom = $] -1; +\infty[$

$\Leftrightarrow 2^{2x} - \frac{1}{2} \cdot 2^x - \frac{1}{9} = 0$ (2)

Posons $y = 2^x > 0$. (2) devient $y^2 - \frac{1}{2}y - \frac{1}{9} = 0$ $\Delta = \frac{25}{36}$

$y = \frac{2}{3}$ ou $y = -\frac{1}{6}$ (à rejeter, car $-\frac{1}{6} < 0$)

Ainsi $2^x = \frac{2}{3}$, donc $x = \log_2 \frac{2}{3} \in \text{dom}$ ($\log_2 \frac{2}{3} \approx -0,58$).

$S = \{1 - \log_2 3\}$

Question III (5 + 5 + 2 + 8 = 20 points)

$$f(x) = 1 - x - \ln \frac{x}{x-1}$$

1)

CE et CD: $x - 1 \neq 0$ et $\frac{x}{x-1} > 0$

$$\text{dom } f = \text{dom}_d f =]-\infty; 0[\cup]1; +\infty[$$

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \left(\underbrace{1-x}_{\rightarrow \mp\infty} - \ln \underbrace{\frac{x}{x-1}}_{\substack{\rightarrow 1 \\ \rightarrow 0}} \right) = \mp\infty$$

$$\text{AO} \equiv y = -x + 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \left(\underbrace{1-x}_{\rightarrow 0} - \ln \underbrace{\frac{\tilde{x}}{x-1}}_{\substack{\rightarrow 1 \\ \rightarrow 0^+ \\ \rightarrow +\infty}} \right) = -\infty$$

$$\text{AVD} \equiv x = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\underbrace{1-x}_{\rightarrow 1} - \ln \underbrace{\frac{\tilde{x}}{x-1}}_{\substack{\rightarrow 0^- \\ \rightarrow -1 \\ \rightarrow 0^+ \\ \rightarrow -\infty}} \right) = +\infty$$

$$\text{AVG} \equiv x = 0$$

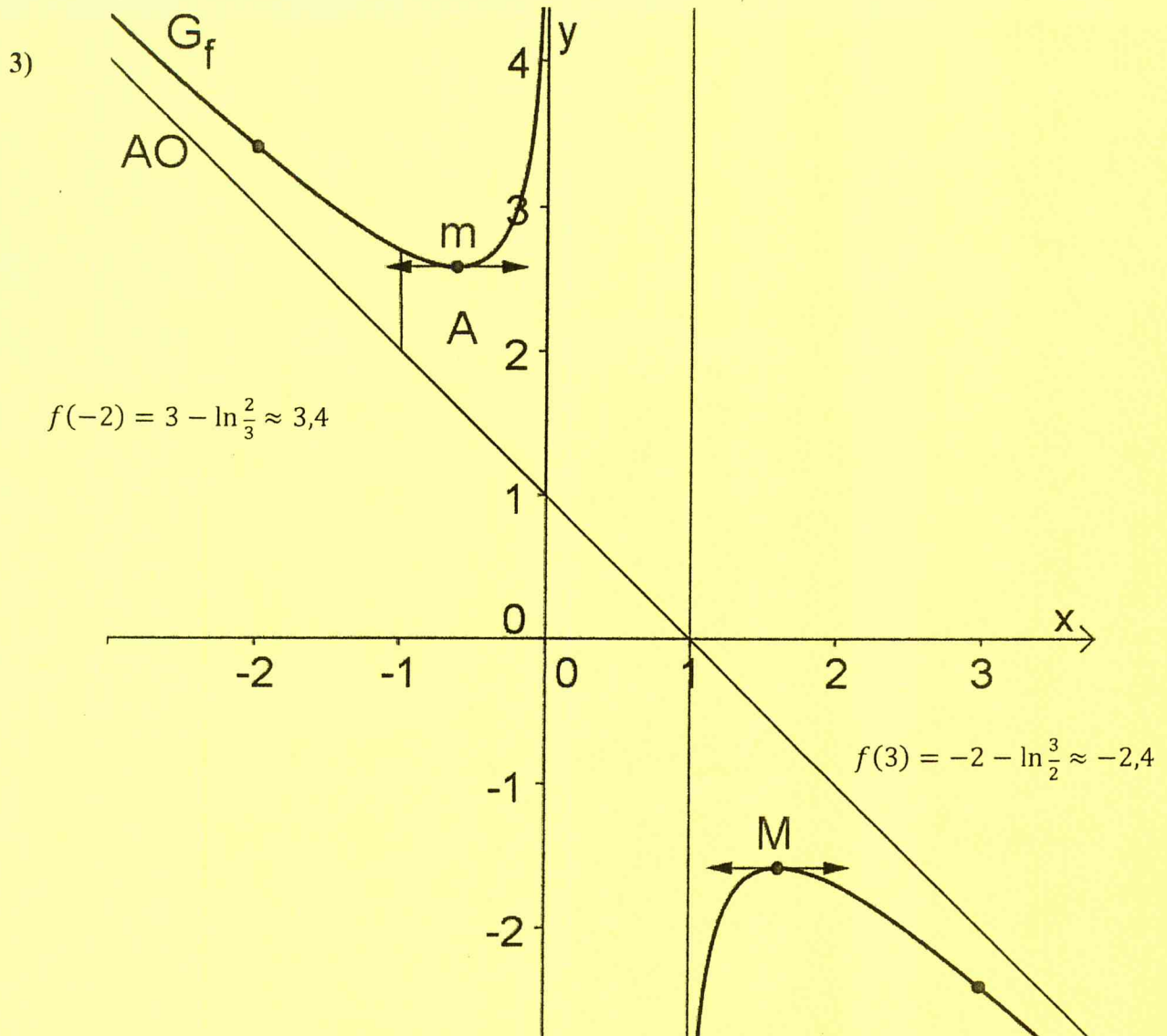
2)

$$f'(x) = -1 - \frac{x-1}{x} \cdot \frac{(x-1) - x}{(x-1)^2} = -1 + \frac{1}{x(x-1)} = \frac{-x^2 + x + 1}{x(x-1)} \quad x_{1/2} = \frac{-1 \pm \sqrt{5}}{-2} < \begin{matrix} \approx -0,6 \\ \approx 1,6 \end{matrix}$$

$$f''(x) = \left(-1 + \frac{1}{x(x-1)} \right)' = 0 - \frac{2x-1}{x^2(x-1)^2} = \frac{1-2x}{x^2(x-1)^2}$$

x	$-\infty$	$\frac{1-\sqrt{5}}{2}$	0	$\frac{1}{2}$	1	$\frac{1+\sqrt{5}}{2}$	$+\infty$			
$f''(x)$		+	+	0	-	-				
$f'(x)$	-	0	+		+	0	-			
$f(x)$	$+\infty$	$\searrow \approx 2,6$	$\nearrow +\infty$		$-\infty$	$\nearrow \approx -1,6$	$\searrow -\infty$			
G_f	AO	\cup	m	\cup	AVG	AVD	\cap	M	\cap	AO

$$\left(f\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2} - \ln \frac{3-\sqrt{5}}{2} \approx 2,58 \text{ et } f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2} - \ln \frac{3+\sqrt{5}}{2} \approx -1,58 \right)$$



4) $\forall x \in]-\infty; 0[\quad F(x) = \int [f(x) - (1-x)] dx = \int -\ln \frac{x}{x-1} dx = \int \ln \frac{x-1}{x} dx$

$u(x) = \ln \frac{x-1}{x}$	$u'(x) = \frac{x}{x-1} \cdot \frac{x-(x-1)}{x^2} = \frac{1}{x(x-1)}$
$v'(x) = 1$	$v(x) = x$

$$F(x) = x \ln \frac{x-1}{x} - \int \frac{1}{x-1} dx = x \ln \frac{x-1}{x} - \ln|x-1| + C = x \ln \frac{x-1}{x} - \ln(1-x) + C \quad (C \in \mathbb{R})$$

$$\forall \lambda \in]-1; 0[\quad A(\lambda) = F(\lambda) - F(-1) = \lambda \ln \frac{\lambda-1}{\lambda} - \ln(1-\lambda) - (-\ln 2 - \ln 2)$$

$$= \lambda \ln \frac{\lambda-1}{\lambda} - \ln(1-\lambda) + 2 \ln 2$$

$$A = \lim_{\lambda \rightarrow 0^-} A(\lambda) = \lim_{\lambda \rightarrow 0^-} \underbrace{\lambda \ln \frac{\lambda-1}{\lambda}}_{=L} - \underbrace{\lim_{\lambda \rightarrow 0^-} \ln(1-\lambda)}_{=\ln 1=0} + 2 \ln 2$$

$$L = \lim_{\lambda \rightarrow 0^-} \underbrace{\lambda}_{\rightarrow 0^-} \underbrace{\ln \frac{\lambda-1}{\lambda}}_{\rightarrow +\infty} \quad f.i. = \lim_{\lambda \rightarrow 0^-} \frac{\ln \frac{\lambda-1}{\lambda}}{\frac{1}{\lambda}} \stackrel{H}{=} \lim_{\lambda \rightarrow 0^-} \frac{1}{\frac{\lambda(\lambda-1)}{-\lambda^2}} = \lim_{\lambda \rightarrow 0^-} \frac{-\lambda}{\lambda-1} = 0$$

$$A = \boxed{2 \ln 2 \text{ u. a.}} \quad (\approx 1,39 \text{ u. a.}; A = 8 \ln 2 \text{ cm}^2 \approx 5,55 \text{ cm}^2)$$

Question IV (3 + 6 + 6 = 15 points)

1)

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(\frac{3x+1}{3x} \right)^{2x-3} &= \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{3x} \right)^{2x-3} \underset{(y=3x)}{=} \lim_{y \rightarrow +\infty} \left(1 + \frac{1}{y} \right)^{\frac{2}{3}y-3} \\ &= \lim_{y \rightarrow +\infty} \underbrace{\left[\left(1 + \frac{1}{y} \right)^y \right]^{\frac{2}{3}}}_{\rightarrow e} \underbrace{\left(1 + \frac{1}{y} \right)^{-3}}_{\rightarrow 1} = \boxed{\sqrt[3]{e^2}} (\approx 1,95) \end{aligned}$$

2)

$$\begin{aligned} \int_{-2}^{2\sqrt{3}} \frac{3x+1}{\sqrt{16-x^2}} dx &= -\frac{3}{2} \int_{-2}^{2\sqrt{3}} \frac{-2x}{\sqrt{16-x^2}} dx + \int_{-2}^{2\sqrt{3}} \frac{\frac{1}{4}}{\sqrt{1-\left(\frac{x}{4}\right)^2}} dx \\ &= -3 \int_{-2}^{2\sqrt{3}} \frac{(16-x^2)'}{2\sqrt{16-x^2}} dx + \int_{-2}^{2\sqrt{3}} \frac{\left(\frac{x}{4}\right)'}{\sqrt{1-\left(\frac{x}{4}\right)^2}} dx \\ &= \left[-3\sqrt{16-x^2} + \arcsin \frac{x}{4} \right]_{-2}^{2\sqrt{3}} \\ &= \left(-6 + \frac{\pi}{3} \right) - \left(-6\sqrt{3} - \frac{\pi}{6} \right) \\ &= \boxed{-6 + \frac{\pi}{2} + 6\sqrt{3}} (\approx 5,96) \end{aligned}$$

3)

$$\begin{aligned} \int \frac{\sin 2x}{(1-2\sin^2 x)^4} dx &= -\frac{1}{2} \int \frac{-4 \sin x \cos x}{(1-2\sin^2 x)^4} dx \\ &= -\frac{1}{2} \int (1-2\sin^2 x)' (1-2\sin^2 x)^{-4} dx \\ &= -\frac{1}{2} \frac{(1-2\sin^2 x)^{-3}}{-3} + C \\ &= \boxed{\frac{1}{6(1-2\sin^2 x)^3} + C} \quad (C \in \mathbb{R}) \end{aligned}$$

$$\text{CE: } 1 - 2\sin^2 x \neq 0 \Leftrightarrow \sin^2 x \neq \frac{1}{2}$$

$$\Leftrightarrow \sin x \neq \frac{\sqrt{2}}{2} \text{ et } \sin x \neq -\frac{\sqrt{2}}{2}$$

$$\Leftrightarrow x \neq \frac{\pi}{4} + k\frac{\pi}{2} \quad (k \in \mathbb{Z})$$

$$\text{p.ex. } \boxed{I = \left] -\frac{\pi}{4}; \frac{\pi}{4} \right[}$$

Question V (8 points)

Soit h la fonction définie par $h(x) = g(x) - f(x) = x^2 - \frac{9}{2}x + 2$. $\Delta = \frac{49}{4}$ $x_1 = \frac{1}{2}$ et $x_2 = 4$

x	$-\infty$	$\frac{1}{2}$	4	$+\infty$
$h(x)$	$+$	0	$-$	0

Comme f est strictement croissante, $f(4) = -1 < 0$ et $h(x) < 0$ pour tout $x \in]\frac{1}{2}, 4[$,
 $g(x) < f(x) < 0$ et $|g(x)| > |f(x)|$ pour tout $x \in]\frac{1}{2}, 4[$.

$$\begin{aligned}
 V &= \pi \int_{\frac{1}{2}}^4 ([g(x)]^2 - [f(x)]^2) dx \\
 &= \pi \int_{\frac{1}{2}}^4 \left[(x^2 - 4x - 1)^2 - \left(\frac{1}{2}x - 3\right)^2 \right] dx \\
 &= \pi \int_{\frac{1}{2}}^4 \left(x^4 + 16x^2 + 1 - 8x^3 - 2x^2 + 8x - \frac{1}{4}x^2 + 3x - 9 \right) dx \\
 &= \pi \int_{\frac{1}{2}}^4 \left(x^4 - 8x^3 + \frac{55}{4}x^2 + 11x - 8 \right) dx \\
 &= \pi \left[\frac{1}{5}x^5 - 2x^4 + \frac{55}{12}x^3 + \frac{11}{2}x^2 - 8x \right]_{\frac{1}{2}}^4 \\
 &= \pi \left[\left(\frac{1024}{5} - 512 + \frac{880}{3} + 88 - 32 \right) - \left(\frac{1}{160} - \frac{1}{8} + \frac{55}{96} + \frac{11}{8} - 4 \right) \right] \\
 &= \pi \left(\frac{632}{15} + \frac{521}{240} \right) \\
 &= \boxed{\frac{10633}{240} \pi \text{ u.v.}} (\approx 139,186 \text{ u.v.})
 \end{aligned}$$

